ELEMENTARY TRIGONOMETRY

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Schoolcraft College
Elementary Trigonometry

Michael Corral

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This is a text on elementary trigonometry, designed for students who have completed courses in high-school algebra and geometry. Though designed for college students, it could also be used in high schools. The traditional topics are covered, but a more geometrical approach is taken than usual. Also, some numerical methods (e.g. the secant method for solving trigonometric equations) are discussed. A brief tutorial on using Gnuplot to graph trigonometric functions is included.

1: RIGHT TRIANGLE TRIGONOMETRY ANGLES

Trigonometry is the study of the relations between the sides and angles of triangles. The word “trigonometry” is derived from the Greek words trigono (τρ´ιγωνο), meaning “triangle”, and metro (μϱτρω´), meaning “measure”. Though the ancient Greeks, such as Hipparchus and Ptolemy, used trigonometry in their study of astronomy between roughly 150 B.C. - A.D. 200, its history is much older.

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2: GENERAL TRIANGLES

Though the methods described will work for right triangles, they are mostly used to solve oblique triangles, that is, triangles which do not have a right angle. There are two types of oblique triangles: an acute triangle has all acute angles, and an obtuse triangle has one obtuse angle. As we will see, Cases 1 and 2 can be solved using the law of sines, Case 3 can be solved using either the law of cosines or the law of tangents, and Case 4 can be solved using the law of cosines.

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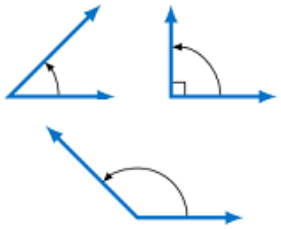
1: RIGHT TRIANGLE TRIGONOMETRY ANGLES

Trigonometry is the study of the relations between the sides and angles of triangles. The word “trigonometry” is derived from the Greek words trigono (τρ΄ιγωνο), meaning “triangle”, and metro (μϱτρω΄), meaning “measure”. Though the ancient Greeks, such as Hipparchus and Ptolemy, used trigonometry in their study of astronomy between roughly 150 B.C. - A.D. 200, its history is much older.



1.1: ANGLES

In elementary geometry, angles are always considered to be positive and not larger than 360°. You also learned that the sum of the angles in a triangle equals 180°, and that an isosceles triangle is a triangle with two sides of equal length. Recall that in a right triangle one of the angles is a right angle. Thus, in a right triangle one of the angles is 90° and the other two angles are acute angles whose sum is 90° (i.e. the other two angles are complementary angles).

1.2: TRIGONOMETRIC FUNCTIONS OF AN ACUTE ANGLE

For a right triangle $\triangle ABC$, with the right angle at C and with lengths a, b, and c. For the acute angle A, call the leg BC its opposite side, and call the leg AC its adjacent side. Recall that the hypotenuse of the triangle is the side AB. The ratios of sides of a right triangle occur often enough in practical applications to warrant their own names, so we can define the six trigonometric functions of A.

1.3: APPLICATIONS AND SOLVING RIGHT TRIANGLES

Throughout its early development, trigonometry was often used as a means of indirect measurement, e.g. determining large distances or lengths by using measurements of angles and small, known distances. Today, trigonometry is widely used in physics, astronomy, engineering, navigation, surveying, and various fields of mathematics and other disciplines. In this section we will see some of the ways in which trigonometry can be applied. Your calculator should be in degree mode for these examples.

1.4: TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

To define the trigonometric functions of any angle - including angles less than 0° or greater than 360° - we need a more general definition of an angle. We say that an angle is formed by rotating a ray OA about the endpoint O (called the vertex), so that the ray is in a new position, denoted by the ray OB. The ray OA is called the initial side of the angle, and OB is the terminal side of the angle.

1.5: ROTATIONS AND REFLECTIONS OF ANGLES

Now that we know how to deal with angles of any measure, we will take a look at how certain geometric operations can help simplify the use of trigonometric functions of any angle, and how some basic relations between those functions can be made. The two operations on which we will concentrate in this section are rotation and reflection.

1.E: RIGHT TRIANGLE TRIGONOMETRY ANGLES (EXERCISES)

These are homework exercises to accompany Corral's "Elementary Trigonometry" Textmap. This is a text on elementary trigonometry, designed for students who have completed courses in high-school algebra and geometry. Though designed for college students, it could also be used in high schools. The traditional topics are covered, but a more geometrical approach is taken than usual. Also, some numerical methods (e.g. the secant method for solving trigonometric equations) are discussed.

1.1: Angles

Recall the following definitions from elementary geometry:

a. An angle is **acute** if it is between 0° and 90°.
b. An angle is a **right angle** if it equals 90°.
c. An angle is **obtuse** if it is between 90° and 180°.
d. An angle is a **straight angle** if it equals 180°.

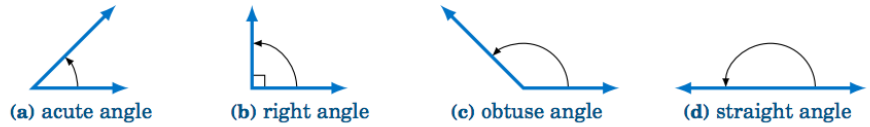


Figure 1.1.1 *Types of angles*

In elementary geometry, angles are always considered to be positive and not larger than 360°. For now we will only consider such angles. The following definitions will be used throughout the text:

a. Two acute angles are **complementary** if their sum equals 90°. In other words, if $0^\circ \leq \angle A, \angle B \leq 90^\circ$ then $\angle A$ and $\angle B$ are complementary if $\angle A + \angle B = 90^\circ$.
b. Two angles between 0° and 180° are **supplementary** if their sum equals 180°. In other words, if $0^\circ \leq \angle A, \angle B \leq 180^\circ$ then $\angle A$ and $\angle B$ are supplementary if $\angle A + \angle B = 180^\circ$.
c. Two angles between 0° and 360° are **conjugate** (or **explementary**) if their sum equals 360°. In other words, if $0^\circ \leq \angle A, \angle B \leq 360^\circ$ then $\angle A$ and $\angle B$ are conjugate if $\angle A + \angle B = 360^\circ$.

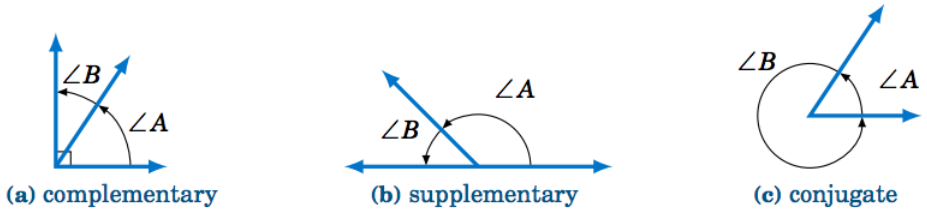


Figure 1.1.2 Types of pairs of angles

Instead of using the angle notation $\angle A$ to denote an angle, we will sometimes use just a capital letter by itself (e.g. A, B, C) or a lowercase variable name (e.g. x, y, t). It is also common to use letters (either uppercase or lowercase) from the Greek alphabet, shown in the table below, to represent angles:

Table 1.1 The Greek alphabet

Letters		Name	Letters		Name	Letters		Name
A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

In elementary geometry you learned that the sum of the angles in a triangle equals 180°, and that an **isosceles triangle** is a triangle with two sides of equal length. Recall that in a **right triangle** one of the angles is a right angle. Thus, in a right

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triangle one of the angles is 90° and the other two angles are acute angles whose sum is 90° (i.e. the other two angles are complementary angles).

$\alpha + 3\alpha + \alpha = 180^\circ \Rightarrow 5\alpha = 180^\circ \Rightarrow \alpha = 36^\circ \Rightarrow \boxed{X = 36^\circ,\ Y = 3 \times 36^\circ = 108^\circ,\ Z = 36^\circ}$

QED

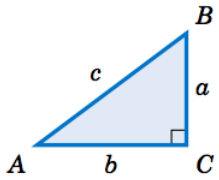

Figure 1.1.4

By knowing the lengths of two sides of a right triangle, the length of the third side can be determined by using the **Pythagorean Theorem**:

The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of its legs.

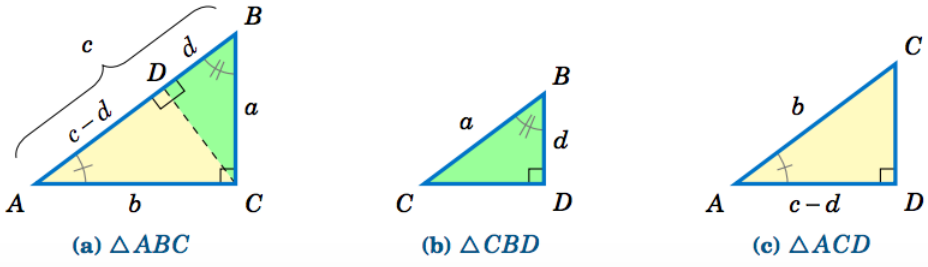

Figure 1.1.5 Similar triangles $\triangle ABC, \triangle CBD, \triangle ACD$

Recall that triangles are **similar** if their corresponding angles are equal, and that similarity implies that corresponding sides are proportional. Thus, since $\triangle ABC$ is similar to $\triangle CBD$, by proportionality of corresponding sides we see that

$$\overline{AB} \text{ is to } \overline{CB} \text{ (hypotenuses) as } \overline{BC} \text{ is to } \overline{BD} \text{ (vertical legs)} \quad \Rightarrow \quad \frac{c}{a} = \frac{a}{d} \quad \Rightarrow \quad cd = a^2 .$$

Since $\triangle ABC$ is similar to $\triangle ACD$, comparing horizontal legs and hypotenuses gives

$$\frac{b}{c-d} = \frac{c}{b} \quad \Rightarrow \quad b^2 = c^2 - cd = c^2 - a^2 \quad \Rightarrow \quad a^2 + b^2 = c^2 . \textbf{ QED}$$

Note: The symbols $\perp$ and $\sim$ denote perpendicularity and similarity, respectively. For example, in the above proof we had $\overline{CD} \perp \overline{AB}$ and $\triangle ABC \sim \triangle CBD \sim \triangle ACD$.

For triangle $\triangle ABC$, the Pythagorean Theorem says that

$$a^2 + 4^2 = 5^2 \quad \Rightarrow \quad a^2 = 25 - 16 = 9 \quad \Rightarrow \quad \boxed{a = 3} .$$

For triangle $\triangle DEF$, the Pythagorean Theorem says that

$$e^2 + 1^2 = 2^2 \quad \Rightarrow \quad e^2 = 4 - 1 = 3 \quad \Rightarrow \quad \boxed{e = \sqrt{3}} .$$

For triangle $\triangle XYZ$, the Pythagorean Theorem says that

$$1^2 + 1^2 = z^2 \quad \Rightarrow \quad z^2 = 2 \quad \Rightarrow \quad \boxed{z = \sqrt{2}}.$$

Let h be the height at which the ladder touches the wall. We can assume that the ground makes a right angle with the wall, as in the picture on the right. Then we see that the ladder, ground, and wall form a right triangle with a hypotenuse of length 17 ft (the length of the ladder) and legs with lengths 8 ft and h ft. So by the Pythagorean Theorem, we have

$$h^2 + 8^2 = 17^2 \quad \Rightarrow \quad h^2 = 289 - 64 = 225 \quad \Rightarrow \quad \boxed{h = 15\,\text{ft}}.$$

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1.2: Trigonometric Functions of an Acute Angle

Consider a right triangle $\triangle ABC$, with the right angle at C and with lengths a, b, and c, as in the figure on the right. For the acute angle A, call the leg $\overline{BC}$ its **opposite side**, and call the leg $\overline{AC}$ its **adjacent side**. Recall that the hypotenuse of the triangle is the side $\overline{AB}$. The ratios of sides of a right triangle occur often enough in practical applications to warrant their own names, so we define the **six trigonometric functions** of A as follows:

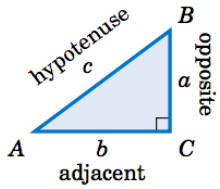


Table 1.2 The six trigonometric functions of A

Name of function	Abbreviation	Definition
sine A	$\sin A$	$= \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c}$
cosine A	$\cos A$	$= \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}$
tangent A	$\tan A$	$= \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b}$
cosecant A	$\csc A$	$= \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{c}{a}$
secant A	$\sec A$	$= \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{c}{b}$
cotangent A	$\cot A$	$= \frac{\text{adjacent side}}{\text{opposite side}} = \frac{b}{a}$

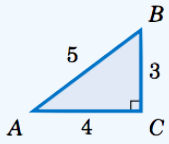
We will usually use the abbreviated names of the functions. Notice from Table 1.2 that the pairs $\sin A$ and $\csc A$, $\cos A$ and $\sec A$, and $\tan A$ and $\cot A$ are reciprocals:

$$\csc A = \frac{1}{\sin A} \qquad \sec A = \frac{1}{\cos A} \qquad \cot A = \frac{1}{\tan A}$$

$$\sin A = \frac{1}{\csc A} \qquad \cos A = \frac{1}{\sec A} \qquad \tan A = \frac{1}{\cot A}$$

Example 1.5

For the right triangle $\triangle ABC$ shown on the right, find the values of all six trigonometric functions of the acute angles A and B.



Solution:

The hypotenuse of $\triangle ABC$ has length 5. For angle A, the opposite side $\overline{BC}$ has length 3 and the adjacent side $\overline{AC}$ has length 4. Thus:

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3} \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4} \qquad \cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}$$

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For angle B, the opposite side $\overline{AC}$ has length 4 and the adjacent side $\overline{BC}$ has length 3. Thus:

$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} \qquad \cos B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} \qquad \tan B = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3} \tag{1.2.1}$$

$$\csc B = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{4} \qquad \sec B = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{3} \qquad \cot B = \frac{\text{adjacent}}{\text{opposite}} = \frac{3}{4} \tag{1.2.2}$$

Notice in Example 1.5 that we did not specify the units for the lengths. This raises the possibility that our answers depended on a triangle of a specific physical size.

For example, suppose that two different students are reading this textbook: one in the United States and one in Germany. The American student thinks that the lengths 3, 4, and 5 in Example 1.5 are measured in inches, while the German student thinks that they are measured in centimeters. Since 1 in ≈ 2.54 cm, the students are using triangles of different physical sizes (see Figure 1.2.1 below, not drawn to scale).

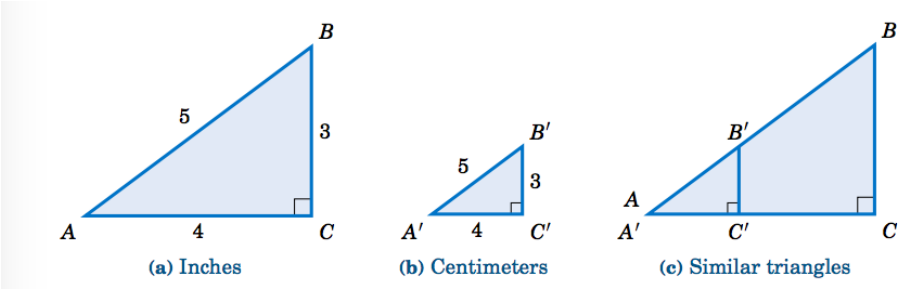


Figure 1.2.1: $\triangle ABC \sim \triangle A'B'C'$

If the American triangle is $\triangle ABC$ and the German triangle is $\triangle A'B'C'$, then we see from Figure 1.2.1 that $\triangle ABC$ is similar to $\triangle A'B'C'$, and hence the corresponding angles are equal and the ratios of the corresponding sides are equal. In fact, we know that common ratio: the sides of $\triangle ABC$ are approximately 2.54 times longer than the corresponding sides of $\triangle A'B'C'$. So when the American student calculates $\sin A$ and the German student calculates $\sin A'$, they get the same answer:

$$\triangle ABC \sim \triangle A'B'C' \quad \Rightarrow \quad \frac{BC}{B'C'} = \frac{AB}{A'B'} \quad \Rightarrow \quad \frac{BC}{AB} = \frac{B'C'}{A'B'} \quad \Rightarrow \quad \sin A = \sin A' \tag{1.2.3}$$

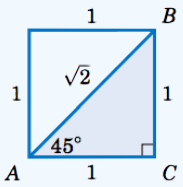
Likewise, the other values of the trigonometric functions of A and A' are the same. In fact, our argument was general enough to work with any similar right triangles. This leads us to the following conclusion:

When calculating the trigonometric functions of an acute angle A, you may use *any* right triangle which has A as one of the angles.

Since we defined the trigonometric functions in terms of ratios of sides, you can think of the units of measurement for those sides as canceling out in those ratios. This means that *the values of the trigonometric functions are unitless numbers*. So when the American student calculated $3/5$ as the value of $\sin A$ in Example 1.5, that is the same as the $3/5$ that the German student calculated, despite the different units for the lengths of the sides.

Example 1.6

Find the values of all six trigonometric functions of 45°.



Solution:

Since we may use any right triangle which has 45° as one of the angles, use the simplest one: take a square whose sides are all 1 unit long and divide it in half diagonally, as in the figure on the right. Since the two legs of the triangle $\triangle ABC$ have the same length, $\triangle ABC$ is an isosceles triangle, which means that the angles A and B are equal. So since $A + B = 90^\circ$, this means that we must have $A = B = 45^\circ$. By the Pythagorean Theorem, the length c of the hypotenuse is given by

$$c^2 = 1^2 + 1^2 = 2 \quad \Rightarrow \quad c = \sqrt{2} \tag{1.2.4}$$

Thus, using the angle A we get:

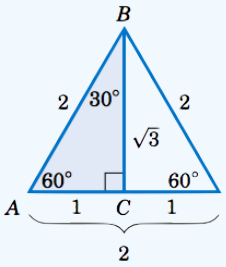
$$\sin 45^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \cos 45^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \tan 45^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1 \tag{1.2.5}$$

$$\csc 45^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = \sqrt{2} \qquad \sec 45^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = \sqrt{2} \qquad \cot 45^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{1} = 1 \tag{1.2.6}$$

Note that we would have obtained the same answers if we had used any right triangle similar to $\triangle ABC$. For example, if we multiply each side of $\triangle ABC$ by $\sqrt{2}$, then we would have a similar triangle with legs of length $\sqrt{2}$ and hypotenuse of length 2. This would give us $\sin 45^\circ = \frac{\sqrt{2}}{2}$, which equals $\frac{\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{\sqrt{2}}$ as before. The same goes for the other functions.

Example 1.7

Find the values of all six trigonometric functions of 60°.



Solution:

Since we may use any right triangle which has 60° as one of the angles, we will use a simple one: take a triangle whose sides are all 2 units long and divide it in half by drawing the bisector from one vertex to the opposite side, as in the figure on the right. Since the original triangle was an *equilateral triangle* (i.e. all three sides had the same length), its three angles were all the same, namely 60°. Recall from elementary geometry that the bisector from the vertex angle of an equilateral triangle to its opposite side bisects both the vertex angle and the opposite side. So as in the figure on the right, the triangle $\triangle ABC$ has angle $A = 60^\circ$ and angle $B = 30^\circ$, which forces the angle C to be 90°. Thus, $\triangle ABC$ is a right triangle. We see that the hypotenuse has length $c = AB = 2$ and the leg $\overline{AC}$ has length $b = AC = 1$. By the Pythagorean Theorem, the length a of the leg $\overline{BC}$ is given by

$$a^2 + b^2 = c^2 \quad \Rightarrow \quad a^2 = 2^2 - 1^2 = 3 \quad \Rightarrow \quad a = \sqrt{3}\,. \tag{1.2.7}$$

Thus, using the angle A we get:

$$\sin 60^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2} \qquad \tan 60^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3} \tag{1.2.8}$$

$$\csc 60^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{2}{\sqrt{3}} \qquad \sec 60^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = 2 \qquad \cot 60^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\sqrt{3}} \tag{1.2.9}$$

Notice that, as a bonus, we get the values of all six trigonometric functions of 30°, by using angle $B = 30^\circ$ in the same triangle $\triangle ABC$ above:

$$\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} \qquad \cos 30^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \tan 30^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}} \tag{1.2.10}$$

$$\csc 30^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = 2 \qquad \sec 30^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{2}{\sqrt{3}} \qquad \cot 30^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{3}}{1} = \sqrt{3} \tag{1.2.11}$$

Example 1.8

A is an acute angle such that $\sin A = \frac{2}{3}$. Find the values of the other trigonometric functions of A.

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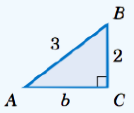


Solution:

In general it helps to draw a right triangle to solve problems of this type. The reason is that the trigonometric functions were defined in terms of ratios of sides of a right triangle, and you are given one such function (the sine, in this case) already in terms of a ratio: $\sin\ A = \frac{2}{3}$. Since $\sin\ A$ is defined as $\frac{\text{opposite}}{\text{hypotenuse}}$, use 2 as the length of the side opposite A and use 3 as the length of the hypotenuse in a right triangle $\triangle ABC$ (see the figure above), so that $\sin\ A = \frac{2}{3}$. The adjacent side to A has unknown length b, but we can use the Pythagorean Theorem to find it:

$$2^2 + b^2 = 3^2 \quad \Rightarrow \quad b^2 = 9 - 4 = 5 \quad \Rightarrow \quad b = \sqrt{5} \tag{1.2.12}$$

We now know the lengths of all sides of the triangle $\triangle ABC$, so we have:

$$\cos\ A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{5}}{3} \qquad \tan\ A = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{\sqrt{5}} \tag{1.2.13}$$

$$\csc\ A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{3}{2} \qquad \sec\ A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{3}{\sqrt{5}} \qquad \cot\ A = \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{5}}{2} \tag{1.2.14}$$

You may have noticed the connections between the sine and cosine, secant and cosecant, and tangent and cotangent of the complementary angles in Examples 1.5 and 1.7. Generalizing those examples gives us the following theorem:

Theorem 1.2 Cofunction Theorem

If A and B are the complementary acute angles in a right triangle $\triangle ABC$, then the following relations hold:

$$\sin\ A = \cos\ B \qquad \sec\ A = \csc\ B \qquad \tan\ A = \cot\ B \tag{1.2.15}$$

$$\sin\ B = \cos\ A \qquad \sec\ B = \csc\ A \qquad \tan\ B = \cot\ A \tag{1.2.16}$$

We say that the pairs of functions $\{\ \sin, \cos\ \}$, $\{\ \sec, \csc\ \}$, and $\{\ \tan, \cot\ \}$ are **cofunctions**

So sine and cosine are cofunctions, secant and cosecant are cofunctions, and tangent and cotangent are cofunctions. That is how the functions cosine, cosecant, and cotangent got the "co" in their names. The Cofunction Theorem says that any trigonometric function of an acute angle is equal to its cofunction of the complementary angle.

Example 1.9

Write each of the following numbers as trigonometric functions of an angle less than 45° : **(a)** $\sin\ 65^\circ$; **(b)** $\cos\ 78^\circ$; **(c)** $\tan\ 59^\circ$.

Solution

(a) The complement of 65° is $90^\circ - 65^\circ = 25^\circ$ and the cofunction of sin is cos, so by the Cofunction Theorem we know that $\sin\ 65^\circ = \cos\ 25^\circ$.

(b) The complement of 78° is $90^\circ - 78^\circ = 12^\circ$ and the cofunction of cos is sin, so $\cos\ 78^\circ = \sin\ 12^\circ$.

(c) The complement of 59° is $90^\circ - 59^\circ = 31^\circ$ and the cofunction of tan is cot, so $\tan\ 59^\circ = \cot\ 31^\circ$.

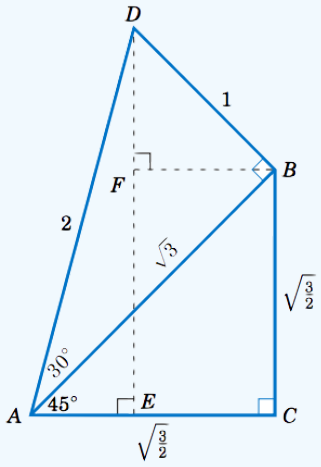
(a) 45−45−90 (b) 30−60−90

Figure 1.2.2 Two general right triangles (any $a > 0$)

The angles 30°, 45°, and 60° arise often in applications. We can use the Pythagorean Theorem to generalize the right triangles in Examples 1.6 and 1.7 and see what *any* $45 - 45 - 90$ and $30 - 60 - 90$ right triangles look like, as in Figure 1.2.2 above.

Example 1.10

Find the sine, cosine, and tangent of 75°.



Solution

Since $75^\circ = 45^\circ + 30^\circ$, place a $30-60-90$ right triangle $\triangle ADB$ with legs of length $\sqrt{3}$ and 1 on top of the hypotenuse of a $45-45-90$ right triangle $\triangle ABC$ whose hypotenuse has length $\sqrt{3}$, as in the figure on the right. From Figure 1.2.2(a) we know that the length of each leg of $\triangle ABC$ is the length of the hypotenuse divided by $\sqrt{2}$. So $AC = BC = \frac{\sqrt{3}}{\sqrt{2}} = \sqrt{\frac{3}{2}}$. Draw $\overline{DE}$ perpendicular to $\overline{AC}$, so that $\triangle ADE$ is a right triangle. Since $\angle BAC = 45^\circ$ and $\angle DAB = 30^\circ$, we see that $\angle DAE = 75^\circ$ since it is the sum of those two angles. Thus, we need to find the sine, cosine, and tangent of $\angle DAE$.

Notice that $\angle ADE = 15^\circ$, since it is the complement of $\angle DAE$. And $\angle ADB = 60^\circ$, since it is the complement of $\angle DAB$. Draw $\overline{BF}$ perpendicular to $\overline{DE}$, so that $\triangle DFB$ is a right triangle. Then $\angle BDF = 45^\circ$, since it is the difference of $\angle ADB = 60^\circ$ and $\angle ADE = 15^\circ$. Also, $\angle DBF = 45^\circ$ since it is the complement of $\angle BDF$. The hypotenuse $\overline{BD}$ of $\triangle DFB$ has length 1 and $\triangle DFB$ is a $45-45-90$ right triangle, so we know that $DF = FB = \frac{1}{\sqrt{2}}$.

Now, we know that $\overline{DE} \perp \overline{AC}$ and $\overline{BC} \perp \overline{AC}$, so $\overline{FE}$ and $\overline{BC}$ are parallel. Likewise, $\overline{FB}$ and $\overline{EC}$ are both perpendicular to $\overline{DE}$ and hence $\overline{FB}$ is parallel to $\overline{EC}$. Thus, $FBCE$ is a rectangle, since $\angle BCE$ is a right angle. So $EC = FB = \frac{1}{\sqrt{2}}$ and $FE = BC = \sqrt{\frac{3}{2}}$. Hence,

$$DE = DF + FE = \frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}} = \frac{\sqrt{3}+1}{\sqrt{2}} \quad \text{and} \quad AE = AC - EC = \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{\sqrt{2}} \text{ . Thus,} \qquad (1.2.17)$$

$$\sin 75^\circ = \frac{DE}{AD} = \frac{\frac{\sqrt{3}+1}{\sqrt{2}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}, \ \cos 75^\circ = \frac{AE}{AD} = \frac{\frac{\sqrt{3}-1}{\sqrt{2}}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}, \text{ and } \tan 75^\circ = \frac{DE}{AE} = \frac{\frac{\sqrt{3}+1}{\sqrt{2}}}{\frac{\sqrt{3}-1}{\sqrt{2}}} = \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} . \qquad (1.2.18)$$

Note: Taking reciprocals, we get $\csc 75^\circ = \frac{4}{\sqrt{6}+\sqrt{2}}$, $\sec 75^\circ = \frac{4}{\sqrt{6}-\sqrt{2}}$, and $\cot 75^\circ = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$.

Contributors and Attributions

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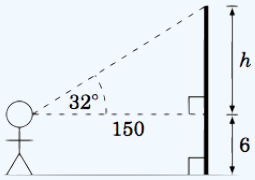
1.3: Applications and Solving Right Triangles

Throughout its early development, trigonometry was often used as a means of indirect measurement, e.g. determining large distances or lengths by using measurements of angles and small, known distances. Today, trigonometry is widely used in physics, astronomy, engineering, navigation, surveying, and various fields of mathematics and other disciplines. In this section we will see some of the ways in which trigonometry can be applied. Your calculator should be in degree mode for these examples.

Example 1.11

A person stands 150 ft away from a flagpole and measures an *angle of elevation* of 32° from his horizontal line of sight to the top of the flagpole. Assume that the person's eyes are a vertical distance of 6 ft from the ground. What is the height of the flagpole?



Solution:

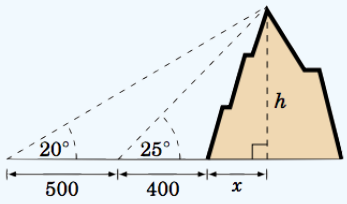
The picture on the right describes the situation. We see that the height of the flagpole is $h+6$ ft, where

$$\frac{h}{150} = \tan 32^\circ \quad \Rightarrow \quad h = 150 \tan 32^\circ = 150\,(0.6249) = 94\,. \tag{1.3.1}$$

How did we know that $\tan 32^\circ = 0.6249$? By using a calculator. And since none of the numbers we were given had decimal places, we rounded off the answer for h to the nearest integer. Thus, the height of the flagpole is $h+6 = 94+6 = \boxed{100 \text{ ft}}$.

Example 1.12

A person standing 400 ft from the base of a mountain measures the angle of elevation from the ground to the top of the mountain to be 25°. The person then walks 500 ft straight back and measures the angle of elevation to now be 20°. How tall is the mountain?



Solution:

We will assume that the ground is flat and not inclined relative to the base of the mountain. Let h be the height of the mountain, and let x be the distance from the base of the mountain to the point directly beneath the top of the mountain, as in the picture on the right. Then we see that

$$\frac{h}{x+400} = \tan 25^\circ \quad \Rightarrow \quad h = (x+400)\tan 25^\circ\,, \text{ and} \tag{1.3.2}$$

$$\frac{h}{x+400+500} = \tan 20^\circ \quad \Rightarrow \quad h = (x+900)\tan 20^\circ\,, \text{ so} \tag{1.3.3}$$

$(x+400)\tan 25^\circ = (x+900)\tan 20^\circ$, since they both equal h. Use that equation to solve for x:

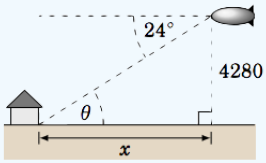
$$x\tan 25^\circ - x\tan 20^\circ = 900\tan 20^\circ - 400\tan 25^\circ \quad \Rightarrow \quad x = \frac{900\tan 20^\circ - 400\tan 25^\circ}{\tan 25^\circ - \tan 20^\circ} = 1378 \text{ ft} \tag{1.3.4}$$

Finally, substitute x into the first formula for h to get the height of the mountain:

$$h = (1378+400)\tan 25^\circ = 1778\,(0.4663) = \boxed{829 \text{ ft}} \tag{1.3.5}$$

Example 1.13

A blimp 4280 ft above the ground measures an *angle of depression* of 24° from its horizontal line of sight to the base of a house on the ground. Assuming the ground is flat, how far away along the ground is the house from the blimp?



Solution:

Let x be the distance along the ground from the blimp to the house, as in the picture to the right. Since the ground and the blimp's horizontal line of sight are parallel, we know from elementary geometry that the angle of elevation θ from the base of the house to the blimp is equal to the angle of depression from the blimp to the base of the house, i.e. $\theta = 24^\circ$. Hence,

$$\frac{4280}{x} = \tan 24^\circ \quad \Rightarrow \quad x = \frac{4280}{\tan 24^\circ} = \boxed{9613 \text{ ft}}. \tag{1.3.6}$$

Example 1.14

An observer at the top of a mountain 3 miles above sea level measures an angle of depression of 2.23° to the ocean horizon. Use this to estimate the radius of the earth.

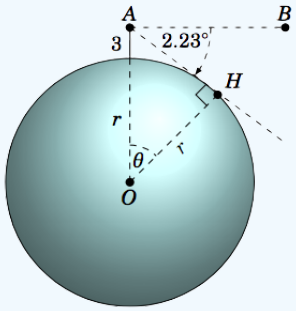


Figure 1.3.1

Solution:

We will assume that the earth is a sphere. Let r be the radius of the earth. Let the point A represent the top of the mountain, and let H be the ocean horizon in the line of sight from A, as in Figure 1.3.1. Let O be the center of the earth, and let B be a point on the horizontal line of sight from A (i.e. on the line perpendicular to $\overline{OA}$). Let θ be the angle $\angle AOH$.

Since A is 3 miles above sea level, we have $OA = r + 3$. Also, $OH = r$. Now since $\overline{AB} \perp \overline{OA}$, we have $\angle OAB = 90^\circ$, so we see that $\angle OAH = 90^\circ - 2.23^\circ = 87.77^\circ$. We see that the line through A and H is a tangent line to the surface of the earth (considering the surface as the circle of radius r through H as in the picture). So by Exercise 14 in Section 1.1, $\overline{AH} \perp \overline{OH}$ and hence $\angle OHA = 90^\circ$. Since the angles in the triangle $\triangle OAH$ add up to 180°, we have $\theta = 180^\circ - 90^\circ - 87.77^\circ = 2.23^\circ$. Thus,

$$\cos\,\theta = \frac{OH}{OA} = \frac{r}{r+3} \quad \Rightarrow \quad \frac{r}{r+3} = \cos\,2.23^\circ, \tag{1.3.7}$$

so solving for r we get

$$r = (r+3)\,\cos\,2.23^\circ \quad \Rightarrow \quad r - r\,\cos\,2.23^\circ = 3\,\cos\,2.23^\circ \tag{1.3.8}$$

$$\Rightarrow \quad r = \frac{3\,\cos\,2.23^\circ}{1 - \cos\,2.23^\circ} \tag{1.3.9}$$

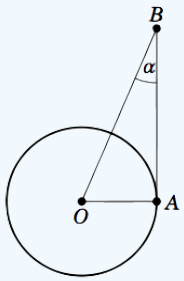
$$\Rightarrow \quad \boxed{r = 3958.3 \text{ miles}}. \tag{1.3.10}$$

Note: This answer is very close to the earth's actual (mean) radius of 3956.6miles.

Example 1.15

As another application of trigonometry to astronomy, we will find the distance from the earth to the sun. Let O be the center of the earth, let A be a point on the equator, and let B represent an object (e.g. a star) in space, as in the picture on the right. If the earth is positioned in such a way that the angle $\angle OAB = 90^\circ$, then we say that the angle $\alpha = \angle OBA$ is the *equatorial parallax* of the object. The equatorial

parallax of the sun has been observed to be approximately $\alpha = 0.00244^\circ$. Use this to estimate the distance from the center of the earth to the sun.



Solution:

Let B be the position of the sun. We want to find the length of $\overline{OB}$. We will use the actual radius of the earth, mentioned at the end of Example 1.14, to get $OA = 3956.6$ miles. Since $\angle OAB = 90^\circ$, we have

$$\frac{OA}{OB} = \sin\alpha \quad \Rightarrow \quad OB = \frac{OA}{\sin\alpha} = \frac{3956.6}{\sin 0.00244^\circ} = 92908394\,, \tag{1.3.11}$$

so the distance from the center of the earth to the sun is approximately $\boxed{93 \text{ million miles}}$.

Note: The earth's orbit around the sun is an ellipse, so the actual distance to the sun varies.

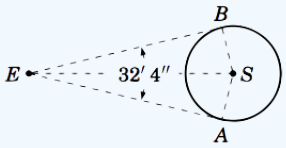
In the above example we used a very small angle (0.00244°). A degree can be divided into smaller units: a **minute** is one-sixtieth of a degree, and a **second** is one-sixtieth of a minute. The symbol for a minute is $'$ and the symbol for a second is $''$. For example, $4.5^\circ = 4^\circ\ 30'$. And $4.505^\circ = 4^\circ\ 30'\ 18''$:

$$4^\circ\ 30'\ 18'' = 4 + \frac{30}{60} + \frac{18}{3600} \text{ degrees} = 4.505^\circ \tag{1.3.12}$$

In Example 1.15 we used $\alpha = 0.00244^\circ \approx 8.8''$, which we mention only because some angle measurement devices do use minutes and seconds.

Example 1.16

An observer on earth measures an angle of $32'\ 4''$ from one visible edge of the sun to the other (opposite) edge, as in the picture on the right. Use this to estimate the radius of the sun.



Solution:

Let the point E be the earth and let S be the center of the sun. The observer's lines of sight to the visible edges of the sun are tangent lines to the sun's surface at the points A and B. Thus, $\angle EAS = \angle EBS = 90^\circ$. The radius of the sun equals AS. Clearly $AS = BS$. So since $EB = EA$ (why?), the triangles $\triangle EAS$ and $\triangle EBS$ are similar. Thus, $\angle AES = \angle BES = \frac{1}{2}\angle AEB = \frac{1}{2}\,(32'\ 4'') = 16'\ 2'' = (16/60) + (2/3600) = 0.26722^\circ$.

Now, ES is the distance from the *surface* of the earth (where the observer stands) to the center of the sun. In Example 1.15 we found the distance from the *center* of the earth to the sun to be $92,908,394$miles. Since we treated the sun in that example as a point, then we are justified in treating that distance as the distance between the centers of the earth and sun. So $ES = 92908394 -$ radius of earth $= 92908394 - 3956.6 = 92904437.4$miles. Hence,

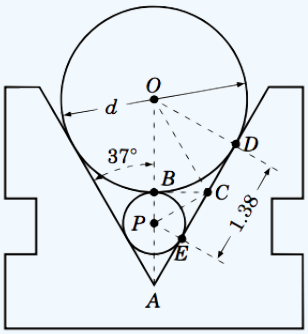
$$\sin(\angle AES) = \frac{AS}{ES} \quad \Rightarrow \quad AS = ES\,\sin 0.26722^\circ = (92904437.4)\,\sin 0.26722^\circ = \boxed{433,293 \text{ miles}}\,. \tag{1.3.13}$$

Note: This answer is close to the sun's actual (mean) radius of $432,200$miles.

You may have noticed that the solutions to the examples we have shown required at least one right triangle. In applied problems it is not always obvious which right triangle to use, which is why these sorts of problems can be difficult. Often no right triangle will be immediately evident, so you will have to create one. There is no general strategy for this, but remember that a right triangle requires a right angle, so look for places where you can form perpendicular line segments. When the problem contains a circle, you can create right angles by using the perpendicularity of the tangent line to the circle at a point with the line that joins that point to the center of the circle. We did exactly that in Examples 1.14, 1.15, and 1.16.

Example 1.17

The machine tool diagram on the right shows a symmetric *V-block*, in which one circular roller sits on top of a smaller circular roller. Each roller touches both slanted sides of the V-block. Find the diameter d of the large roller, given the information in the diagram.



Solution:

The diameter d of the large roller is twice the radius OB, so we need to find OB. To do this, we will show that $\triangle OBC$ is a right triangle, then find the angle $\angle BOC$, and then find BC. The length OB will then be simple to determine.

Since the slanted sides are tangent to each roller, $\angle ODA = \angle PEC = 90^\circ$. By symmetry, since the vertical line through the centers of the rollers makes a 37° angle with each slanted side, we have $\angle OAD = 37^\circ$. Hence, since $\triangle ODA$ is a right triangle, $\angle DOA$ is the complement of $\angle OAD$. So $\angle DOA = 53^\circ$.

Since the horizontal line segment $\overline{BC}$ is tangent to each roller, $\angle OBC = \angle PBC = 90^\circ$. Thus, $\triangle OBC$ is a right triangle. And since $\angle ODA = 90^\circ$, we know that $\triangle ODC$ is a right triangle. Now, $OB = OD$ (since they each equal the radius of the large roller), so by the Pythagorean Theorem we have $BC = DC$:

$$BC^2 = OC^2 - OB^2 = OC^2 - OD^2 = DC^2 \quad \Rightarrow \quad BC = DC \tag{1.3.14}$$

Thus, $\triangle OBC$ and $\triangle ODC$ are *congruent triangles* (which we denote by $\triangle OBC \cong \triangle ODC$), since their corresponding sides are equal. Thus, their corresponding angles are equal. So in particular, $\angle BOC = \angle DOC$. We know that $\angle DOB = \angle DOA = 53^\circ$. Thus,

$$53^\circ = \angle DOB = \angle BOC + \angle DOC = \angle BOC + \angle BOC = 2\angle BOC \quad \Rightarrow \quad \angle BOC = 26.5^\circ. \tag{1.3.15}$$

Likewise, since $BP = EP$ and $\angle PBC = \angle PEC = 90^\circ$, $\triangle BPC$ and $\triangle EPC$ are congruent right triangles. Thus, $BC = EC$. But we know that $BC = DC$, and we see from the diagram that $EC + DC = 1.38$. Thus, $BC + BC = 1.38$ and so $BC = 0.69$. We now have all we need to find OB:

$$\frac{BC}{OB} = \tan\angle BOC \quad \Rightarrow \quad OB = \frac{BC}{\tan\angle BOC} = \frac{0.69}{\tan 26.5^\circ} = 1.384 \tag{1.3.16}$$

Hence, the diameter of the large roller is $d = 2 \times OB = 2\,(1.384) = \boxed{2.768}$ ~.

Example 1.18

A *slider-crank mechanism* is shown in Figure 1.3.2 below. As the piston moves downward the connecting rod rotates the crank in the clockwise direction, as indicated.

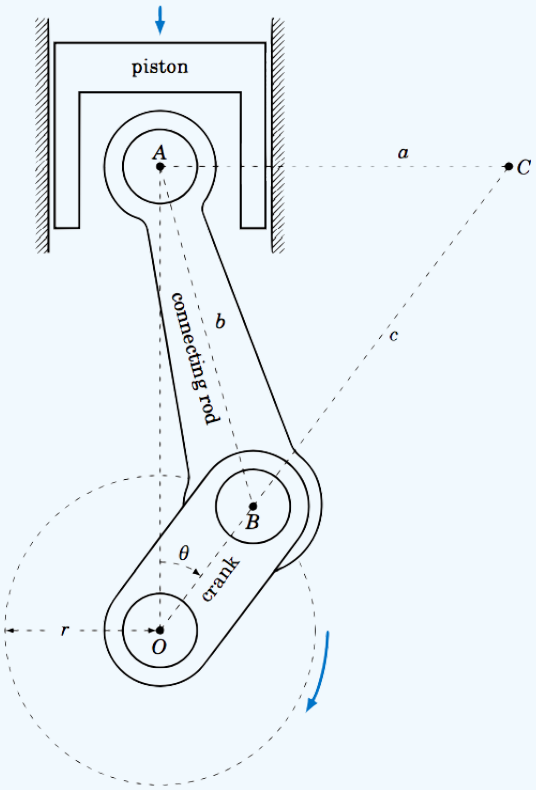
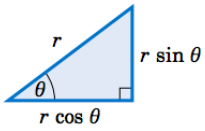


Figure 1.3.2 Slider-crank mechanism

The point A is the center of the connecting rod's *wrist pin* and only moves vertically. The point B is the center of the *crank pin* and moves around a circle of radius r centered at the point O, which is directly below A and does not move. As the crank rotates it makes an angle θ with the line $\overline{OA}$. The *instantaneous center of rotation* of the connecting rod at a given time is the point C where the horizontal line through A intersects the extended line through O and B. From Figure 1.3.2 we see that $\angle OAC = 90^\circ$, and we let $a = AC$, $b = AB$, and $c = BC$. In the exercises you will show that for $0^\circ < \theta < 90^\circ$,

$$c = \frac{\sqrt{b^2 - r^2\,(\sin\theta)^2}}{\cos\theta} \qquad \text{and} \qquad a = r\,\sin\theta + \sqrt{b^2 - r^2\,(\sin\theta)^2}\,\tan\theta\,. \tag{1.3.17}$$

For some problems it may help to remember that when a right triangle has a hypotenuse of length r and an acute angle θ, as in the picture below, the adjacent side will have length $r\cos\theta$ and the opposite side will have length $r\sin\theta$. You can think of those lengths as the horizontal and vertical ``components'' of the hypotenuse.



Notice in the above right triangle that we were given two pieces of information: one of the acute angles and the length of the hypotenuse. From this we determined the lengths of the other two sides, and the other acute angle is just the complement of the known acute angle. In general, a triangle has six parts: three sides and three angles. **Solving a triangle** means finding the unknown parts based on the known parts. In the case of a right triangle, one part is always known: one of the angles is 90°.

Example 1.19

Solve the right triangle in Figure 1.3.3 using the given information:

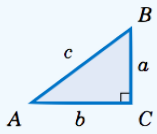


Figure 1.3.3

(a) $c = 10,\ A = 22^\circ$
Solution: The unknown parts are a, b, and B. Solving yields:

$$a = c \sin A = 10 \sin 22^\circ = 3.75 \quad (1.3.18)$$
$$b = c \cos A = 10 \cos 22^\circ = 9.27 \quad (1.3.19)$$
$$B = 90^\circ - A = 90^\circ - 22^\circ = 68^\circ \quad (1.3.20)$$

(b) $b = 8,\ A = 40^\circ$
Solution: The unknown parts are a, c, and B. Solving yields:

$$\frac{a}{b} = \tan A \quad \Rightarrow \quad a = b \tan A = 8 \tan 40^\circ = 6.71 \quad (1.3.21)$$
$$\frac{b}{c} = \cos A \quad \Rightarrow \quad c = \frac{b}{\cos A} = \frac{8}{\cos 40^\circ} = 10.44 \quad (1.3.22)$$

(c) $a = 3,\ b = 4$
Solution: The unknown parts are c, A, and B. By the Pythagorean Theorem,

$$c = \sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5\,. \quad (1.3.23)$$

Now, $\tan A = \frac{a}{b} = \frac{3}{4} = 0.75$. So how do we find A? There should be a key labeled $\boxed{\tan^{-1}}$ on your calculator, which works like this: give it a number x and it will tell you the angle θ such that $\tan \theta = x$. In our case, we want the angle A such that $\tan A = 0.75$:

$$\text{Enter: } 0.75 \quad \text{Press: } \boxed{\tan^{-1}} \quad \text{Answer: } 36.86989765 \quad (1.3.24)$$

This tells us that $A = 36.87^\circ$, approximately. Thus $B = 90^\circ - A = 90^\circ - 36.87^\circ = 53.13^\circ$.

Note: The $\boxed{\sin^{-1}}$ and $\boxed{\cos^{-1}}$ keys work similarly for sine and cosine, respectively. These keys use the *inverse trigonometric functions*, which we will discuss in Chapter 5.

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1.4: Trigonometric Functions of Any Angle

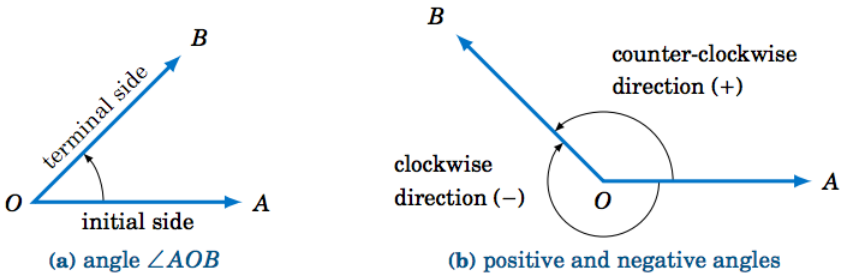


Figure 1.4.1 Definition of a general angle

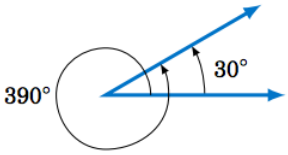


Figure 1.4.2 Angle greater than 360°

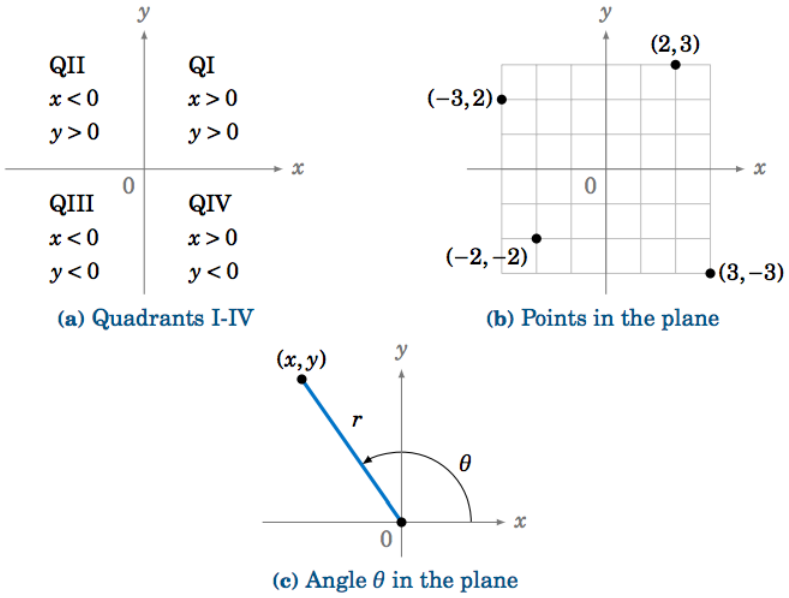


Figure 1.4.3 xy-coordinate plane

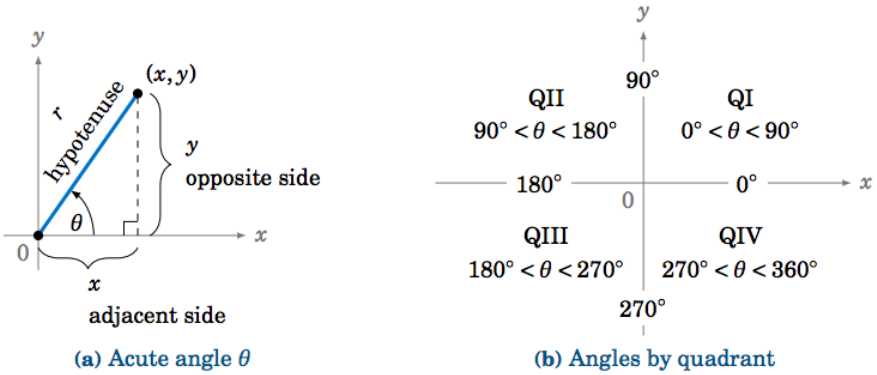
Now let θ be any angle. We say that θ is in **standard position** if its initial side is the positive x-axis and its vertex is the origin $(0,0)$. Pick *any* point (x,y) on the terminal side of θ a distance $r>0$ from the origin (see Figure 1.4.3(c)). (Note that $r=\sqrt{x^2+y^2}$. Why?) We then define the trigonometric functions of θ as follows:

$$\sin\,\theta=\frac{y}{r} \qquad \cos\,\theta=\frac{x}{r} \qquad \tan\,\theta=\frac{y}{x} \tag{1.4.1}$$

$$\csc\,\theta=\frac{r}{y} \qquad \sec\,\theta=\frac{r}{x} \qquad \cot\,\theta=\frac{x}{y} \tag{1.4.2}$$

As in the acute case, by the use of similar triangles these definitions are well-defined (i.e. they do not depend on which point (x,y) we choose on the terminal side of θ). Also, notice that $|\sin\,\theta|\le 1$ and $|\cos\,\theta|\le 1$, since $|y|\le r$ and $|x|\le r$ in the above definitions.

Notice that in the case of an acute angle these definitions are equivalent to our earlier definitions in terms of right triangles: draw a right triangle with angle θ such that $x =$ adjacent side, $y =$ opposite side, and $r =$ hypotenuse. For example, this would give us $\sin\theta = \frac{y}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos\theta = \frac{x}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$, just as before (see Figure 1.4.4(a)).



(a) Acute angle θ (b) Angles by quadrant

Figure 1.4.4

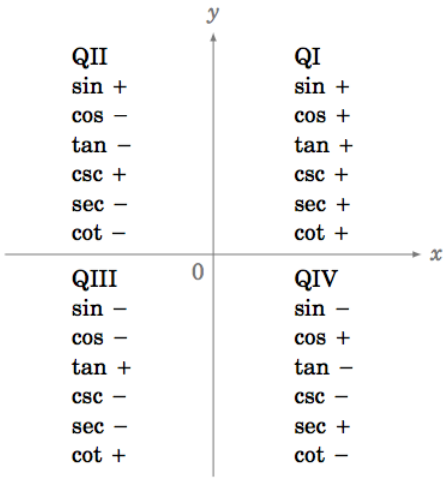


Figure 1.4.5 Signs of the trigonometric functions by quadrant

We know $120° = 180° - 60°$. By Example 1.7 in Section 1.2, we see that we can use the point $(-1, \sqrt{3})$ on the terminal side of the angle $120°$ in QII, since we saw in that example that a basic right triangle with a $60°$ angle has adjacent side of length 1, opposite side of length $\sqrt{3}$, and hypotenuse of length 2, as in the figure on the right. Drawing that triangle in QII so that the hypotenuse is on the terminal side of $120°$ makes $r = 2$, $x = -1$, and $y = \sqrt{3}$. Hence:

$$\sin 120° = \frac{y}{r} = \frac{\sqrt{3}}{2} \qquad \cos 120° = \frac{x}{r} = \frac{-1}{2} \qquad \tan 120° = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\csc 120° = \frac{r}{y} = \frac{2}{\sqrt{3}} \qquad \sec 120° = \frac{r}{x} = \frac{2}{-1} = -2 \qquad \cot 120° = \frac{x}{y} = \frac{-1}{\sqrt{3}}$$

We know that $225° = 180° + 45°$. By Example 1.6 in Section 1.2, we see that we can use the point $(-1, -1)$ on the terminal side of the angle $225°$ in QIII, since we saw in that example that a basic right triangle with a $45°$ angle has adjacent side of length 1, opposite side of length 1, and hypotenuse of length $\sqrt{2}$, as in the figure on the right. Drawing that triangle in QIII so that the hypotenuse is on the terminal side of $225°$ makes $r = \sqrt{2}$, $x = -1$, and $y = -1$. Hence:

$$\sin 225^\circ = \frac{y}{r} = \frac{-1}{\sqrt{2}} \qquad \cos 225^\circ = \frac{x}{r} = \frac{-1}{\sqrt{2}} \qquad \tan 225^\circ = \frac{y}{x} = \frac{-1}{-1} = 1$$

$$\csc 225^\circ = \frac{r}{y} = -\sqrt{2} \qquad \sec 225^\circ = \frac{r}{x} = -\sqrt{2} \qquad \cot 225^\circ = \frac{x}{y} = \frac{-1}{-1} = 1$$

We know that $330^\circ = 360^\circ - 30^\circ$. By Example 1.7 in Section 1.2, we see that we can use the point $(\sqrt{3}, -1)$ on the terminal side of the angle 225° in QIV, since we saw in that example that a basic right triangle with a 30° angle has adjacent side of length $\sqrt{3}$, opposite side of length 1, and hypotenuse of length 2, as in the figure on the right. Drawing that triangle in QIV so that the hypotenuse is on the terminal side of 330° makes $r = 2$, $x = \sqrt{3}$, and $y = -1$. Hence:

$$\sin 330^\circ = \frac{y}{r} = \frac{-1}{2} \qquad \cos 330^\circ = \frac{x}{r} = \frac{\sqrt{3}}{2} \qquad \tan 330^\circ = \frac{y}{x} = \frac{-1}{\sqrt{3}}$$

$$\csc 330^\circ = \frac{r}{y} = -2 \qquad \sec 330^\circ = \frac{r}{x} = \frac{2}{\sqrt{3}} \qquad \cot 330^\circ = \frac{x}{y} = -\sqrt{3}$$

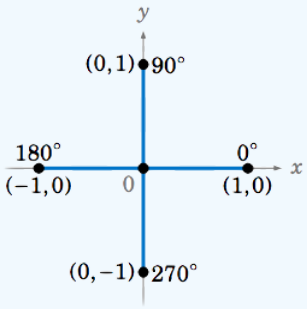


Figure 1.4.6

Solution:

These angles are different from the angles we have considered so far, in that the terminal sides lie along either the x-axis or the y-axis. So unlike the previous examples, we do not have any right triangles to draw. However, the values of the trigonometric functions are easy to calculate by picking the simplest points on their terminal sides and then using the definitions in formulas Equation 1.4.1 and Equation 1.4.2.

For instance, for the angle 0° use the point $(1, 0)$ on its terminal side (the positive x-axis), as in Figure 1.4.6. You could think of the line segment from the origin to the point $(1, 0)$ as sort of a degenerate right triangle whose height is 0 and whose hypotenuse and base have the same length 1. Regardless, in the formulas we would use $r = 1$, $x = 1$, and $y = 0$. Hence:

$$\sin 0^\circ = \frac{y}{r} = \frac{0}{1} = 0 \qquad \cos 0^\circ = \frac{x}{r} = \frac{1}{1} = 1 \qquad \tan 0^\circ = \frac{y}{x} = \frac{0}{1} = 0$$

$$\csc 0^\circ = \frac{r}{y} = \frac{1}{0} = \text{undefined} \qquad \sec 0^\circ = \frac{r}{x} = \frac{1}{1} = 1 \qquad \cot 0^\circ = \frac{x}{y} = \frac{1}{0} = \text{undefined}$$

Note that $\csc\ 0^\circ$ and $\cot\ 0^\circ$ are undefined, since division by 0 is not allowed.

Similarly, from Figure 1.4.6 we see that for 90° the terminal side is the positive y-axis, so use the point $(0, 1)$. Again, you could think of the line segment from the origin to $(0, 1)$ as a degenerate right triangle whose base has length 0 and whose height equals the length of the hypotenuse. We have $r = 1$, $x = 0$, and $y = 1$, and hence:

$$\sin 90^\circ = \frac{y}{r} = \frac{1}{1} = 1 \qquad \cos 90^\circ = \frac{x}{r} = \frac{0}{1} = 0 \qquad \tan 90^\circ = \frac{y}{x} = \frac{1}{0} = \text{undefined}$$

$$\csc 90^\circ = \frac{r}{y} = \frac{1}{1} = 1 \qquad \sec 90^\circ = \frac{r}{x} = \frac{1}{0} = \text{undefined} \qquad \cot 90^\circ = \frac{x}{y} = \frac{0}{1} = 0$$

Likewise, for 180° use the point $(-1, 0)$ so that $r = 1$, $x = -1$, and $y = 0$. Hence:

$$\sin 180^\circ = \frac{y}{r} = \frac{0}{1} = 0 \qquad \cos 180^\circ = \frac{x}{r} = \frac{-1}{1} = -1 \qquad \tan 180^\circ = \frac{y}{x} = \frac{0}{-1} = 0$$

$$\csc 180^\circ = \frac{r}{y} = \frac{1}{0} = \text{undefined} \qquad \sec 180^\circ = \frac{r}{x} = \frac{1}{-1} = -1$$

$$\cot 180^\circ = \frac{x}{y} = \frac{-1}{0} = \text{undefined}$$

Lastly, for 270° use the point $(0, -1)$ so that $r = 1$, $x = 0$, and $y = -1$. Hence:

$$\sin 270^\circ = \frac{y}{r} = \frac{-1}{1} = -1 \qquad \cos 270^\circ = \frac{x}{r} = \frac{0}{1} = 0 \qquad \tan 270^\circ = \frac{y}{x} = \frac{-1}{0} = \text{undefined}$$

$$\csc 270^\circ = \frac{r}{y} = \frac{1}{-1} = -1 \qquad \sec 270^\circ = \frac{r}{x} = \frac{1}{0} = \text{undefined} \qquad \cot 270^\circ = \frac{x}{y} = \frac{0}{-1} = 0$$

The following table summarizes the values of the trigonometric functions of angles between 0° and 360° which are integer multiples of 30° or 45°:

Table 1.3 Table of trigonometric function values

Angle	sin	cos	tan	csc	sec	cot
0°	0	1	0	undefined	1	undefined
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$
90°	1	0	undefined	1	undefined	0
120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2}{\sqrt{3}}$	-2	$-\frac{1}{\sqrt{3}}$
135°	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
150°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	2	$-\frac{2}{\sqrt{3}}$	$-\sqrt{3}$
180°	0	-1	0	undefined	-1	undefined
210°	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	-2	$-\frac{2}{\sqrt{3}}$	$\sqrt{3}$
225°	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
240°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	-2	$\frac{1}{\sqrt{3}}$
270°	-1	0	undefined	-1	undefined	0
300°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	2	$-\frac{1}{\sqrt{3}}$
315°	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
330°	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	-2	$\frac{2}{\sqrt{3}}$	$-\sqrt{3}$

Since 360° represents one full revolution, the trigonometric function values repeat every 360°. For example, $\sin 360^\circ = \sin 0^\circ$, $\cos 390^\circ = \cos 30^\circ$, $\tan 540^\circ = \tan 180^\circ$, $\sin(-45^\circ) = \sin 315^\circ$, etc. In general, if two angles differ by an integer multiple of 360° then each trigonometric function will have equal values at both angles. Angles such as these, which have the same initial and terminal sides, are called **coterminal**.

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In Examples 1.20-1.22, we saw how the values of trigonometric functions of an angle θ larger than 90° were found by using a certain acute angle as part of a right triangle. That acute angle has a special name: if θ is a nonacute angle then we say that the **reference angle** for θ is the acute angle formed by the terminal side of θ and either the positive or negative x-axis. So in Example 1.20, we see that 60° is the reference angle for the nonacute angle $\theta = 120^\circ$; in Example 1.21, 45° is the reference angle for $\theta = 225^\circ$; and in Example 1.22, 30° is the reference angle for $\theta = 330^\circ$.

Example 1.24

Let $\theta = 928^\circ$.

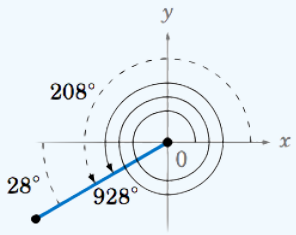

Figure 1.4.7

a. Which angle between 0° and 360° has the same terminal side (and hence the same trigonometric function values) as θ?

b. What is the reference angle for θ?

Solution

(a) Since $928^\circ = 2 \times 360^\circ + 208^\circ$, then θ has the same terminal side as 208°, as in Figure 1.4.7.

(b) 928° and 208° have the same terminal side in QIII, so the reference angle for $\theta = 928^\circ$ is $208^\circ - 180^\circ = 28^\circ$.

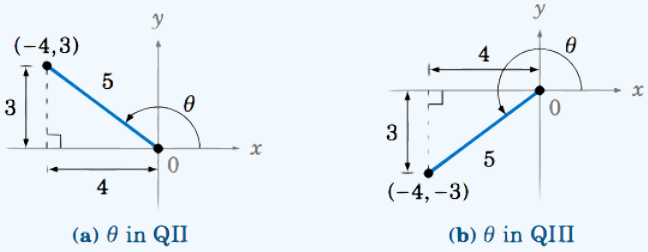

Figure 1.4.8 $\cos\theta = -\frac{4}{5}$

When θ is in QII, we see from Figure 1.4.8(a) that the point $(-4, 3)$ is on the terminal side of θ, and so we have $x = -4$, $y = 3$, and $r = 5$. Thus, $\sin\theta = \frac{y}{r} = \frac{3}{5}$ and $\tan\theta = \frac{y}{x} = \frac{3}{-4}$.

When θ is in QIII, we see from Figure 1.4.8(b) that the point $(-4, -3)$ is on the terminal side of θ, and so we have $x = -4$, $y = -3$, and $r = 5$. Thus, $\sin\theta = \frac{y}{r} = \frac{-3}{5}$ and $\tan\theta = \frac{y}{x} = \frac{-3}{-4} = \frac{3}{4}$.

Thus, either $\boxed{\sin\theta = \frac{3}{5} \text{ and } \tan\theta = -\frac{3}{4}}$ or $\boxed{\sin\theta = -\frac{3}{5} \text{ and } \tan\theta = \frac{3}{4}}$.

Since reciprocals have the same sign, $\csc\theta$ and $\sin\theta$ have the same sign, $\sec\theta$ and $\cos\theta$ have the same sign, and $\cot\theta$ and $\tan\theta$ have the same sign. So it suffices to remember the signs of $\sin\theta$, $\cos\theta$, and $\tan\theta$:

For an angle θ in standard position and a point (x, y) on its terminal side:

a. $\sin\theta$ has the same sign as y
b. $\cos\theta$ has the same sign as x
c. $\tan\theta$ is positive when x and y have the same sign
d. $\tan\theta$ is negative when x and y have opposite signs

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1.5: Rotations and Reflections of Angles

Now that we know how to deal with angles of any measure, we will take a look at how certain geometric operations can help simplify the use of trigonometric functions of any angle, and how some basic relations between those functions can be made. The two operations on which we will concentrate in this section are *rotation* and *reflection*.

To **rotate an angle** means to rotate its terminal side around the origin when the angle is in standard position. For example, suppose we rotate an angle θ around the origin by 90° in the counterclockwise direction. In Figure 1.5.1 we see an angle θ in QI which is rotated by 90°, resulting in the angle $\theta+90^\circ$ in QII. Notice that the complement of θ in the right triangle in QI is the same as the supplement of the angle $\theta+90^\circ$ in QII, since the sum of θ, its complement, and 90° equals 180°. This forces the other angle of the right triangle in QII to be θ.

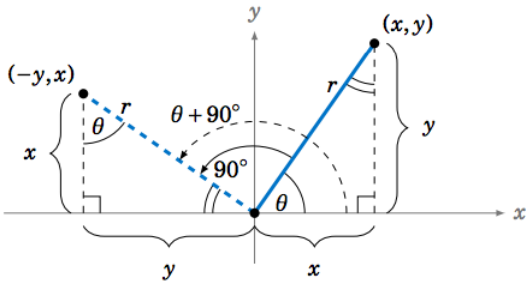


Figure 1.5.1 Rotation of an angle θ by 90°

Thus, the right triangle in QI is similar to the right triangle in QII, since the triangles have the same angles. The rotation of θ by 90° does not change the length r of its terminal side, so the hypotenuses of the similar right triangles are equal, and hence by similarity the remaining corresponding sides are also equal. Using Figure 1.5.1 to match up those corresponding sides shows that the point $(-y, x)$ is on the terminal side of $\theta+90^\circ$ when (x, y) is on the terminal side of θ. Hence, by definition,

$$\sin(\theta+90^\circ) = \frac{x}{r} = \cos\theta, \quad \cos(\theta+90^\circ) = \frac{-y}{r} = -\sin\theta, \quad \tan(\theta+90^\circ) = \frac{x}{-y} = -\cot\theta.$$

Though we showed this for θ in QI, it is easy (see Exercise 4) to use similar arguments for the other quadrants. In general, the following relations hold for all angles θ:

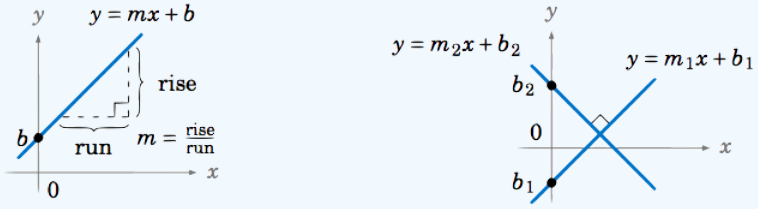
$$\sin(\theta+90^\circ) = \cos\theta \tag{1.5.1}$$

$$\cos(\theta+90^\circ) = -\sin\theta \tag{1.5.2}$$

$$\tan(\theta+90^\circ) = -\cot\theta \tag{1.5.3}$$

Example 1.26

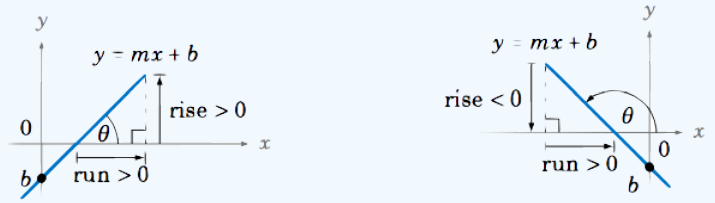
Recall that any nonvertical line in the xy-coordinate plane can be written as $y = mx + b$, where m is the *slope* of the line (defined as $m = \frac{\text{rise}}{\text{run}}$) and b is the y-*intercept*}, i.e. where the line crosses the y-axis (see Figure 1.5.2(a)). We will show that the slopes of perpendicular lines are negative reciprocals. That is, if $y = m_1x + b_1$ and $y = m_2x + b_2$ are nonvertical and nonhorizontal perpendicular lines, then $m_2 = -\frac{1}{m_1}$ (see Figure 1.5.2(b)).



(a) Slope of a line (b) Perpendicular lines

Figure 1.5.2

First, suppose that a line $y = mx + b$ has nonzero slope. The line crosses the x-axis somewhere, so let θ be the angle that the positive x-axis makes with the part of the line above the x-axis, as in Figure 1.5.3. For $m > 0$ we see that θ is acute and $\tan\ \theta = \frac{\text{rise}}{\text{run}} = m$.



(a) $m > 0$, θ acute, rise > 0 (b) $m < 0$, θ obtuse, rise < 0

Figure 1.5.3

If $m < 0$, then we see that θ is obtuse and the rise is negative. Since the run is always positive, our definition of $\tan\ \theta$ from Section 1.4 means that $\tan\ \theta = \frac{-\text{rise}}{-\text{run}} = \frac{\text{rise}}{\text{run}} = m$ (just imagine in Figure 1.5.3(b) the entire line being shifted horizontally to go through the origin, so that θ is unchanged and the point $(-\text{run}, -\text{rise})$ is on the terminal side of θ). Hence:

> *For a line $y = mx + b$ with $m \neq 0$, the slope is given by $m = \tan\ \theta$, where θ is the angle formed by the positive x-axis and the part of the line above the x-axis.}*

Now, in Figure 1.5.2(b) we see that if two lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are perpendicular then rotating one line counterclockwise by 90° around the point of intersection gives us the second line. So if θ is the angle that the line $y = m_1x + b_1$ makes with the positive x-axis, then $\theta + 90^\circ$ is the angle that the line $y = m_2x + b_2$ makes with the positive x-axis. So by what we just showed, $m_1 = \tan\ \theta$ and $m_2 = \tan\ (\theta + 90^\circ)$. But by formula Equation 1.5.3 we know that $\tan\ (\theta + 90^\circ) = -\cot\ \theta$. Hence, $m_2 = -\cot\ \theta = -\frac{1}{\tan\ \theta} = -\frac{1}{m_1}$. **QED**

Rotating an angle θ by 90° in the clockwise direction results in the angle $\theta - 90^\circ$. We could use another geometric argument to derive trigonometric relations involving $\theta - 90^\circ$, but it is easier to use a simple trick: since Equations 1.5.1-1.5.3 hold for *any* angle θ, just replace θ by $\theta - 90^\circ$ in each formula. Since $(\theta - 90^\circ) + 90^\circ = \theta$, this gives us:

$$\sin\ (\theta - 90^\circ) = -\cos\ \theta \tag{1.5.4}$$

$$\cos\ (\theta - 90^\circ) = \sin\ \theta \tag{1.5.5}$$

$$\tan\ (\theta - 90^\circ) = -\cot\ \theta \tag{1.5.6}$$

We now consider rotating an angle θ by 180°. Notice from Figure 1.5.4 that the angles $\theta \pm 180^\circ$ have the same terminal side, and are in the quadrant opposite θ.

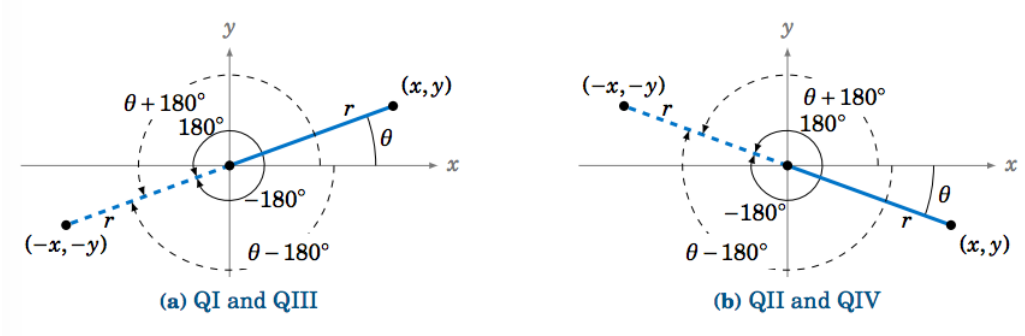


Figure 1.5.4 Rotation of θ by $\pm 180^\circ$

Since $(-x, -y)$ is on the terminal side of $\theta \pm 180^\circ$ when (x, y) is on the terminal side of θ, we get the following relations, which hold for all θ:

$$\sin(\theta \pm 180^\circ) = -\sin\theta \tag{1.5.7}$$

$$\cos(\theta \pm 180^\circ) = -\cos\theta \tag{1.5.8}$$

$$\tan(\theta \pm 180^\circ) = \tan\theta \tag{1.5.9}$$

A **reflection** is simply the mirror image of an object. For example, in Figure 1.5.5 the original object is in QI, its reflection around the y-axis is in QII, and its reflection around the x-axis is in QIV. Notice that if we first reflect the object in QI around the y-axis and then follow that with a reflection around the x-axis, we get an image in QIII. That image is the *reflection around the origin* of the original object, and it is equivalent to a rotation of 180° around the origin. Notice also that a reflection around the y-axis is equivalent to a reflection around the x-axis followed by a rotation of 180° around the origin.

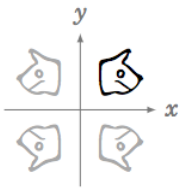


Figure 1.5.5

Applying this to angles, we see that the reflection of an angle θ around the x-axis is the angle $-\theta$, as in Figure 1.5.6.

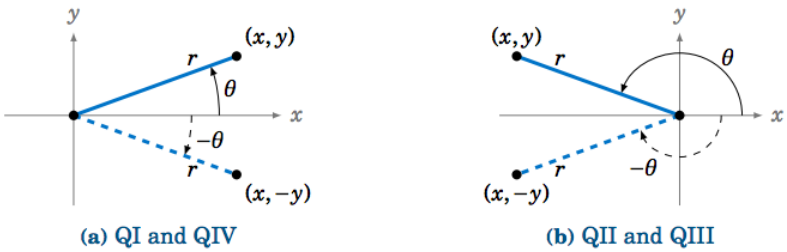


Figure 1.5.6 Reflection of θ around the x-axis

So we see that reflecting a point (x, y) around the x-axis just replaces y by $-y$. Hence:

$$\sin(-\theta) = -\sin\theta \tag{1.5.10}$$

$$\cos(-\theta) = \cos\theta \tag{1.5.11}$$

$$\tan(-\theta) = -\tan\theta \tag{1.5.12}$$

Notice that the cosine function does not change in Equation 1.5.11 because it depends on x, and not on y, for a point (x, y) on the terminal side of θ.

In general, a function $f(x)$ is an **even function** if $f(-x) = f(x)$ for all x, and it is called an **odd function** if $f(-x) = -f(x)$ for all x. Thus, the cosine function is even, while the sine and tangent functions are odd.

Replacing θ by $-\theta$ in Equations 1.5.1-1.5.3, then using Equations 1.5.10-1.5.12, gives:

$$\sin(90^\circ - \theta) = \cos\theta \tag{1.5.13}$$
$$\cos(90^\circ - \theta) = \sin\theta \tag{1.5.14}$$
$$\tan(90^\circ - \theta) = \cot\theta \tag{1.5.15}$$

Note that Equations 1.5.13-1.5.15 extend the Cofunction Theorem from Section 1.2 to *all* θ, not just acute angles. Similarly, Equations 1.5.7-1.5.9 and 1.5.10-1.5.12 give:

$$\sin(180^\circ - \theta) = \sin\theta \tag{1.5.16}$$
$$\cos(180^\circ - \theta) = -\cos\theta \tag{1.5.17}$$
$$\tan(180^\circ - \theta) = -\tan\theta \tag{1.5.18}$$

Notice that reflection around the y-axis is equivalent to reflection around the x-axis ($\theta \mapsto -\theta$) followed by a rotation of 180° ($-\theta \mapsto -\theta + 180^\circ = 180^\circ - \theta$), as in Figure 1.5.7.

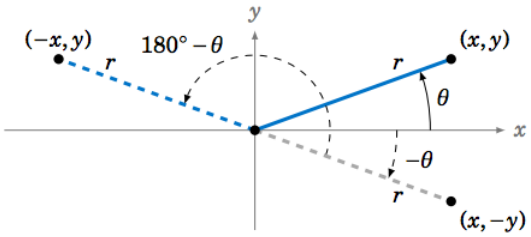


Figure 1.5.7 Reflection of θ around the y-axis = $180^\circ - \theta$

It may seem that these geometrical operations and formulas are not necessary for evaluating the trigonometric functions, since we could just use a calculator. However, there are two reasons for why they are useful. First, the formulas work for any angles, so they are often used to prove general formulas in mathematics and other fields, as we will see later in the text. Second, they can help in determining which angles have a given trigonometric function value.

Example 1.27

Find all angles $0^\circ \le \theta < 360^\circ$ such that $\sin\theta = -0.682$.

Solution

Using the $\boxed{\sin^{-1}}$ button on a calculator with -0.682 as the input, we get $\theta = -43^\circ$, which is not between 0° and 360°. Since $\theta = -43^\circ$ is in QIV, its reflection $180^\circ - \theta$ around the y-axis will be in QIII and have the same sine value. But $180^\circ - \theta = 180^\circ - (-43^\circ) = 223^\circ$ (see Figure 1.5.8). Also, we know that -43° and $-43^\circ + 360^\circ = 317^\circ$ have the same trigonometric function values. So since angles in QI and QII have positive sine values, we see that the only angles between 0° and 360° with a sine of -0.682 are $\boxed{\theta = 223^\circ \text{ and } 317^\circ}$.

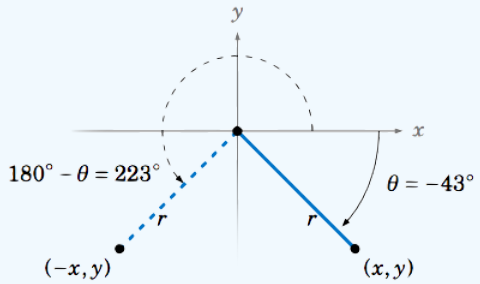


Figure 1.5.8 Reflection around the y-axis: -43° and 223°

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1.E: Right Triangle Trigonometry Angles (Exercises)

These are homework exercises to accompany Corral's "Elementary Trigonometry" Textmap. This is a text on elementary trigonometry, designed for students who have completed courses in high-school algebra and geometry. Though designed for college students, it could also be used in high schools. The traditional topics are covered, but a more geometrical approach is taken than usual. Also, some numerical methods (e.g. the secant method for solving trigonometric equations) are discussed.

1.1 Exercises

For Exercises 1-4, find the numeric value of the indicated angle(s) for the triangle $\triangle ABC$.

1.1.1 Find B if $A = 15^\circ$ and $C = 50^\circ$.

1.1.2 Find C if $A = 110^\circ$ and $B = 31^\circ$.

1.1.3 Find A and B if $C = 24^\circ$, $A = \alpha$, and $B = 2\alpha$.

1.1.4 Find A, B, and C if $A = \beta$ and $B = C = 4\beta$.

For Exercises 5-8, find the numeric value of the indicated angle(s) for the right triangle $\triangle ABC$, with C being the right angle.

1.1.5 Find B if $A = 45^\circ$.

1.1.6 Find A and B if $A = \alpha$ and $B = 2\alpha$.

1.1.7 Find A and B if $A = \phi$ and $B = \phi^2$.

1.1.8 Find A and B if $A = \theta$ and $B = 1/\theta$.

1.1.9 A car goes 24 miles due north then 7 miles due east. What is the straight distance between the car's starting point and end point?

1.1.10 One end of a rope is attached to the top of a pole 100 ft high. If the rope is 150 ft long, what is the maximum distance along the ground from the base of the pole to where the other end can be attached? You may assume that the pole is perpendicular to the ground.

1.1.11 Prove that the hypotenuse is the longest side in every right triangle. (*Hint: Is* $a^2 + b^2 > a^2$?)

1.1.12 Can a right triangle have sides with lengths 2, 5, and 6? Explain your answer.

1.1.13 If the lengths a, b, and c of the sides of a right triangle are positive integers, with $a^2 + b^2 = c^2$, then they form what is called a **Pythagorean triple**. The triple is normally written as (a,b,c). For example, (3,4,5) and (5,12,13) are well-known Pythagorean triples.
(a) Show that (6,8,10) is a Pythagorean triple.
(b) Show that if (a,b,c) is a Pythagorean triple then so is (ka,kb,kc) for any integer $k > 0$. How would you interpret this geometrically?
(c) Show that $(2mn, m^2 - n^2, m^2 + n^2)$ is a Pythagorean triple for all integers $m > n > 0$.
(d) The triple in part(c) is known as **Euclid's formula** for generating Pythagorean triples. Write down the first ten Pythagorean triples generated by this formula, i.e. use: $m = 2$ and $n = 1$; $m = 3$ and $n = 1$, 2; $m = 4$ and $n = 1$, 2, 3; $m = 5$ and $n = 1$, 2, 3, 4.

1.1.14 This exercise will describe how to draw a line through any point outside a circle such that the line intersects the circle at

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only one point. This is called a *tangent line* to the circle (see the picture on the left in Figure 1.1.6), a notion which we will use throughout the text.

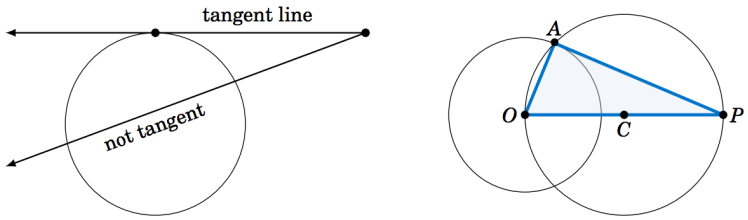
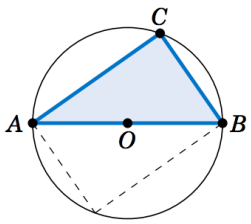


Figure 1.1.6

On a sheet of paper draw a circle of radius 1 inch, and call the center of that circle O. Pick a point P which is 2.5 inches away from O. Draw the circle which has $\overline{OP}$ as a diameter, as in the picture on the right in Figure 1.1.6. Let A be one of the points where this circle intersects the first circle. Draw the line through P and A. In general the tangent line through a point on a circle is perpendicular to the line joining that point to the center of the circle (why?). Use this fact to explain why the line you drew is the tangent line through A and to calculate the length of $\overline{PA}$. Does it match the physical measurement of $\overline{PA}$?

1.1.15 Suppose that $\triangle ABC$ is a triangle with side $\overline{AB}$ a diameter of a circle with center O, as in the picture on the right, and suppose that the vertex C lies on the circle. Now imagine that you rotate the circle 180° around its center, so that $\triangle ABC$ is in a new position, as indicated by the dashed lines in the picture. Explain how this picture proves Thales' Theorem.



1.2 Exercises

For Exercises 1-10, find the values of all six trigonometric functions of\\angles A and B in the right triangle $\triangle ABC$ in Figure 1.2.3.

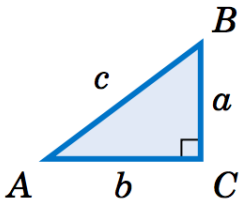


Figure 1.2.3

1.2.1 $a = 5, b = 12, c = 13$

1.2.2 $a = 8, b = 15, c = 17$

1.2.3 $a = 7, b = 24, c = 25$

1.2.4 $a = 20, b = 21, c = 29$

1.2.5 $a = 9, b = 40, c = 41$

1.2.6 $a = 1, b = 2, c = \sqrt{5}$

1.2.7 $a = 1, b = 3$

1.2.8 $a = 2, b = 5$

1.2.9 $a = 5, c = 6$

1.2.10 $b = 7, c = 8$

For Exercises 11-18, find the values of the other five trigonometric functions of the acute angle A given the indicated value of one of the functions.

1.2.11 $\sin A = \frac{3}{4}$

1.2.12 $\cos A = \frac{2}{3}$

1.2.13 $\cos A = \frac{2}{\sqrt{10}}$

1.2.14 $\sin A = \frac{2}{4}$

1.2.15 $\tan A = \frac{5}{9}$

1.2.16 $\tan A = 3$

1.2.17 $\sec A = \frac{7}{3}$

1.2.18 $\csc A = 3$

For Exercises 19-23, write the given number as a trigonometric function of an acute angle less than 45°.

1.2.19 $\sin 87^\circ$

1.2.20 $\sin 53^\circ$

1.2.21 $\cos 46^\circ$

1.2.22 $\tan 66^\circ$

1.2.23 $\sec 77^\circ$

For Exercises 24-28, write the given number as a trigonometric function of an acute angle greater than 45°.

1.2.24 $\sin 1^\circ$

1.2.25 $\cos 13^\circ$

1.2.26 $\tan 26^\circ$

1.2.27 $\cot 10^\circ$

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1.2.28 $\csc\ 43^\circ$

1.2.29 In Example 1.7 we found the values of all six trigonometric functions of 60° and 30°.

(a) Does $\sin\ 30^\circ + \sin\ 30^\circ = \sin\ 60^\circ$?
(b) Does $\cos\ 30^\circ + \cos\ 30^\circ = \cos\ 60^\circ$?
(c) Does $\tan\ 30^\circ + \tan\ 30^\circ = \tan\ 60^\circ$?
(d) Does $2\ \sin\ 30^\circ\ \cos\ 30^\circ = \sin\ 60^\circ$?

1.2.30 For an acute angle A, can $\sin\ A$ be larger than 1? Explain your answer.

1.2.31 For an acute angle A, can $\cos\ A$ be larger than 1? Explain your answer.

1.2.32 For an acute angle A, can $\sin\ A$ be larger than $\tan\ A$? Explain your answer.

1.2.33 If A and B are acute angles and $A < B$, explain why $\sin\ A < \sin\ B$.

1.2.34 If A and B are acute angles and $A < B$, explain why $\cos\ A > \cos\ B$.

1.2.35 Prove the Cofunction Theorem (Theorem 1.2). (*Hint: Draw a right triangle and label the angles and sides.*)

1.2.36 Use Example 1.10 to find all six trigonometric functions of 15°.

1.2.37 In Figure 1.2.4, $\overline{CB}$ is a diameter of a circle with a radius of 2 cm and center O, $\triangle ABC$ is a right triangle, and $\overline{CD}$ has length $\sqrt{3}$ cm.

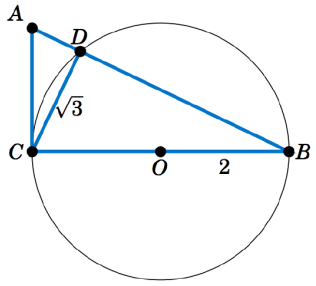


Figure 1.2.4

(a) Find $\sin\ A$. (*Hint: Use Thales' Theorem.*)
(b) Find the length of $\overline{AC}$.
(c) Find the length of $\overline{AD}$.
(d) Figure 1.2.4 is drawn to scale. Use a protractor to measure the angle A, then use your calculator to find\\the sine of that angle. Is the calculator result close to your answer from part(a)? Note: Make sure that your calculator is in degree mode.

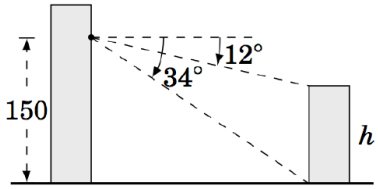
1.2.38 In Exercise 37, verify that the area of $\triangle ABC$ equals $\frac{1}{2}AB \cdot CD$. Why does this make sense?

1.2.39 In Exercise 37, verify that the area of $\triangle ABC$ equals $\frac{1}{2}AB \cdot AC\ \sin\ A$.

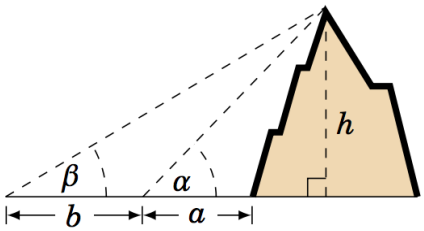
1.2.40 In Exercise 37, verify that the area of $\triangle ABC$ equals $\frac{1}{2}(BC)^2\ \cot\ A$.

1.3 Exercises

1.3.1 From a position 150 ft above the ground, an observer in a building measures angles of depression of 12° and 34° to the top and bottom, respectively, of a smaller building, as in the picture on the right. Use this to find the height h of the smaller building.

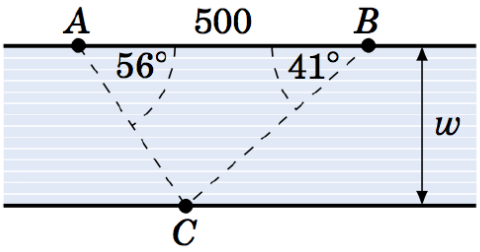


1.3.2 Generalize Example 1.12: A person standing a ft from the base of a mountain measures an angle of elevation α from the ground to the top of the mountain. The person then walks b ft straight back and measures an angle of elevation β to the top of the mountain, as in the picture on the right. Assuming the ground is level, find a formula for the height h of the mountain in terms of a, b, α, and β.



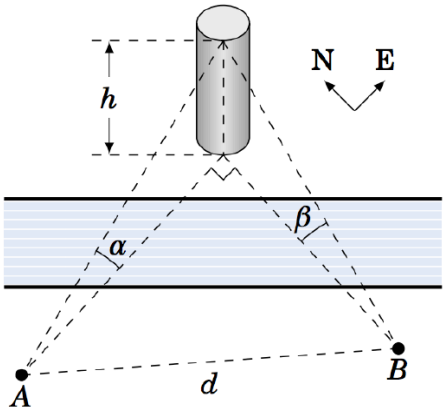
1.3.3 As the angle of elevation from the top of a tower to the sun decreases from 64° to 49° during the day, the length of the shadow of the tower increases by 92 ft along the ground. Assuming the ground is level, find the height of the tower.

1.3.4 Two banks of a river are parallel, and the distance between two points A and B along one bank is 500 ft. For a point C on the opposite bank, $\angle BAC = 56^\circ$ and $\angle ABC = 41^\circ$, as in the picture on the right. What is the width w of the river? (*Hint: Divide $\overline{AB}$ into two pieces.*)



1.3.5 A tower on one side of a river is directly east and north of points A and B, respectively, on the other side of the river. The top of the tower has angles of elevation α and β from A and B, respectively, as in the picture on the right. Let d be the distance between A and B. Assuming that both sides of the river are at the same elevation, show that the height h of the tower is

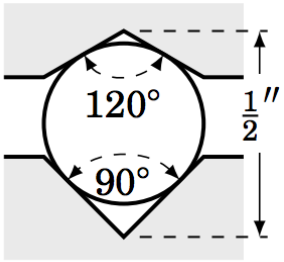
$$h = \frac{d}{\sqrt{(\cot\alpha)^2 + (\cot\beta)^2}}.$$



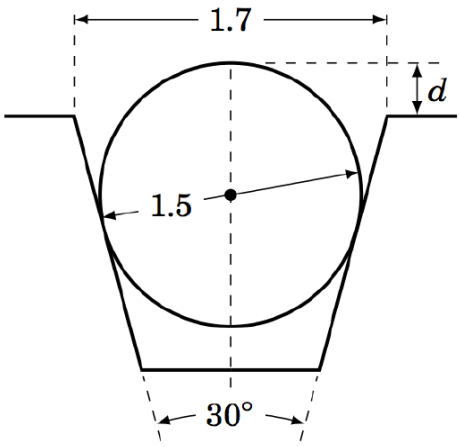
1.3.6 The equatorial parallax of the moon has been observed to be approximately $57'$. Taking the radius of the earth to be 3956.6miles, estimate the distance from the center of
the earth to the moon. (*Hint: See Example* 1.15.)

1.3.7 An observer on earth measures an angle of $31'\ 7''$ from one visible edge of the moon to the other (opposite) edge. Use this to estimate the radius of the moon. (*Hint: Use Exercise*
6 and see Example 1.16.)

1.3.8 A ball bearing sits between two metal grooves, with the top groove having an angle of $120°$ and the bottom groove having an angle of $90°$, as in the picture on the right. What must the diameter of the ball bearing be for the distance between the vertexes of the grooves to be half an inch? You may assume that the top vertex is directly above the bottom vertex.



1.3.9 The machine tool diagram on the right shows a symmetric *worm thread*, in which a circular roller of diameter 1.5 inches sits. Find the amount d that the top of the roller rises above the top of the thread, given the information in the diagram. (*Hint: Extend the slanted sides of the thread until they meet at a point.*)



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1.3.10 Repeat Exercise 9 using 1.8 inches as the distance across the top of the worm thread.

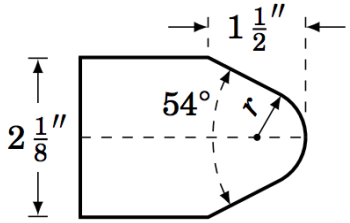
1.3.11 In Exercise 9, what would the distance across the top of the worm thread have to be to make d equal to 0 inches?

1.3.12 For $0^\circ < \theta < 90^\circ$ in the slider-crank mechanism in Example 1.18, show that

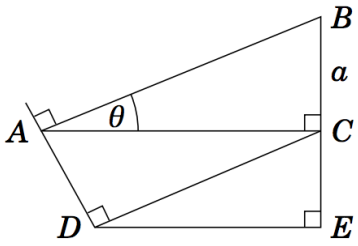
$$c = \frac{\sqrt{b^2 - r^2\,(\sin\theta)^2}}{\cos\theta} \qquad \text{and} \qquad a = r\,\sin\theta + \sqrt{b^2 - r^2\,(\sin\theta)^2}\,\tan\theta\,.$$

(Hint: In Figure 1.3.2 draw line segments from B perpendicular to $\overline{OA}$ and $\overline{AC}$.)

1.3.13 The machine tool diagram on the right shows a symmetric *die punch*. In this view, the rounded tip is part of a circle of radius r, and the slanted sides are tangent to that circle and form an angle of 54°. The top and bottom sides of the die punch are horizontal. Use the
information in the diagram to find the radius r.



1.3.14 In the figure on the right, $\angle BAC = \theta$ and $BC = a$. Use this to find AB, AC, AD, DC, CE, and DE in terms of θ and a.(Hint: What is the angle $\angle ACD$?)



For Exercises 15-23, solve the right triangle in Figure 1.3.4 using the given information.

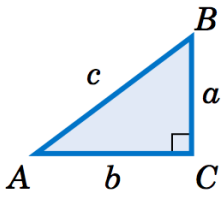


Figure 1.3.4

1.3.15 $a = 5\,,\ b = 12$

1.3.16 $c = 6\,,\ B = 35^\circ$

1.3.17 $b = 2\,,\ A = 8^\circ$

1.3.18 $a = 2\,,\ c = 7$

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1.3.19 $a = 3$, $A = 26^\circ$

1.3.20 $b = 1$, $c = 2$

1.3.21 $b = 3$, $B = 26^\circ$

1.3.22 $a = 2$, $B = 8^\circ$

1.3.23 $c = 2$, $A = 45^\circ$

1.3.24 In Example 1.10 in Section 1.2, we found the exact values of all six trigonometric functions of 75°. For example, we showed that $\cot 75^\circ = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$. So since $\tan 15^\circ = \cot 75^\circ$ by the Cofunction Theorem, this means that $\tan 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$. We will now describe another method for finding the exact values of the trigonometric functions of 15°. In fact, it can be used to find the exact values for the trigonometric functions of $\frac{\theta}{2}$ when those for θ are known, for any $0^\circ < \theta < 90^\circ$. The method is illustrated in Figure 1.3.5 and is described below.

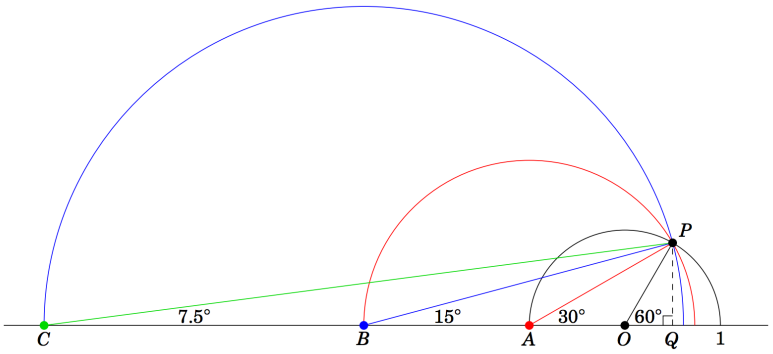

Figure 1.3.5

Draw a semicircle of radius 1 centered at a point O on a horizontal line. Let P be the point on the semicircle such that $\overline{OP}$ makes an angle of 60° with the horizontal line, as in Figure 1.3.5. Draw a line straight down from P to the horizontal line at the point Q. Now create a second semicircle as follows: Let A be the left endpoint of the first semicircle, then draw a new semicircle centered at A with radius equal to AP. Then create a third semicircle in the same way: Let B be the left endpoint of the second semicircle, then draw a new semicircle centered at B with radius equal to BP.

This procedure can be continued indefinitely to create more semicircles. In general, it can be shown that the line segment from the center of the new semicircle to P makes an angle with the horizontal line equal to half the angle from the previous semicircle's center to P.

(a) Explain why $\angle PAQ = 30^\circ$. (*Hint: What is the supplement of* 60°?)
(b) Explain why $\angle PBQ = 15^\circ$ and $\angle PCQ = 7.5^\circ$.
(c) Use Figure 1.3.5 to find the exact values of $\sin 15^\circ$, $\cos 15^\circ$, and $\tan 15^\circ$. (Hint: To start, you will need to use $\angle POQ = 60^\circ$ and $OP = 1$ to find the exact lengths of $\overline{PQ}$ and $\overline{OQ}$.)
(d) Use Figure 1.3.5 to calculate the exact value of $\tan 7.5^\circ$.
(e) Use the same method but with an initial angle of $\angle POQ = 45^\circ$ to find the exact values of $\sin 22.5^\circ$, $\cos 22.5^\circ$, and $\tan 22.5^\circ$.

1.3.25 A manufacturer needs to place ten identical ball bearings against the inner side of a circular container such that each ball bearing touches two other ball bearings, as in the picture on the right. The (inner) radius of the container is 4 cm.

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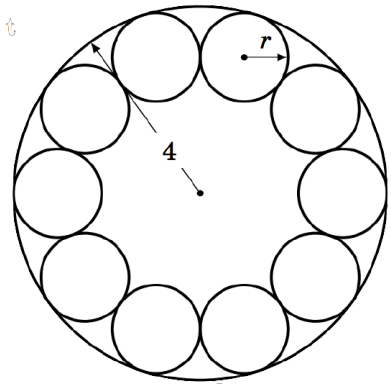
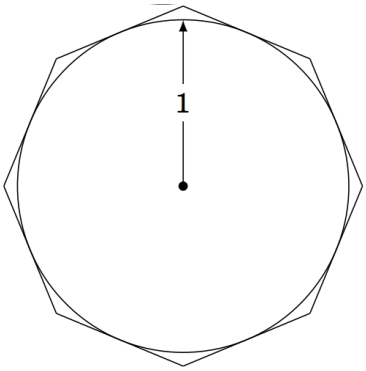


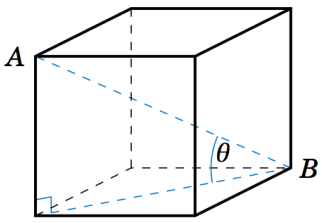
(a) Find the common radius r of the ball bearings.
(b) The manufacturer needs to place a circular ring\\inside the container. What is the largest possible (outer) radius of the ring such that it is not on top\\of the ball bearings and its base is level with the\\base of the container?

1.3.26 A circle of radius 1 is *inscribed* inside a polygon with eight sides of equal length, called a *regular octagon*. That is, each of the eight sides is tangent to the circle, as in the picture on the right.



(a) Calculate the area of the octagon.
(b) If you were to increase the number of sides of the\\polygon, would the area inside it increase or decrease? What number would the area approach, if any? Explain.
(c) Inscribe a regular octagon inside the same circle. That is, draw a regular octagon such that each of its eight vertexes touches the circle. Calculate the area of this octagon.

1.3.27 The picture on the right shows a cube whose sides are of length $a > 0$.



(a) Find the length of the diagonal line segment $\overline{AB}$.
(b) Find the angle θ that $\overline{AB}$ makes with the base of the cube.

1.3.28 In Figure 1.3.6, suppose that α, β, and AD are known. Show that:

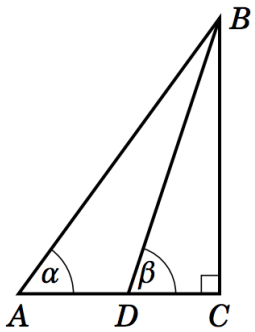


Figure 1.3.6

(a) $BC = \dfrac{AD}{\cot\ \alpha - \cot\ \beta}$

(b) $AC = \dfrac{AD \cdot \tan\ \beta}{\tan\ \beta - \tan\ \alpha}$

(c) $BD = \dfrac{AD \cdot \sin\ \alpha}{\sin\ (\beta - \alpha)}$

(Hint: What is the measure of the angle $\angle\ ABD$?)

1.3.29 Persons A and B are at the beach, their eyes are 5 ft and 6 ft, respectively, above sea level. How many miles farther out is Person B's horizon than Person A's? (Note: 1 mile = 5280 ft)

1.4 Exercises

For Exercises 1-10, state in which quadrant or on which axis the given angle lies.

1.4.1 127°

1.4.2 -127°

1.4.3 313°

1.4.4 -313°

1.4.5 -90°

1.4.6 621°

1.4.7 230°

1.4.8 2009°

1.4.9 1079°

1.4.10 -514°

1.4.11 In which quadrant(s) do sine and cosine have the same sign?

1.4.12 In which quadrant(s) do sine and cosine have the opposite sign?

1.4.13 In which quadrant(s) do sine and tangent have the same sign?

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1.4.14 In which quadrant(s) do sine and tangent have the opposite sign?

1.4.15 In which quadrant(s) do cosine and tangent have the same sign?

1.4.16 In which quadrant(s) do cosine and tangent have the opposite sign?

For Exercises 17-21, find the reference angle for the given angle.

1.4.17 317°

1.4.18 63°

1.4.19 -126°

1.4.20 696°

1.4.21 275°

For Exercises 22-26, find the exact values of $\sin\ \theta$ and $\tan\ \theta$ when $\cos\ \theta$ has the indicated value.

1.4.22 $\cos\ \theta = \frac{1}{2}$

1.4.23 $\cos\ \theta = -\frac{1}{2}$

1.4.24 $\cos\ \theta = 0$

1.4.25 $\cos\ \theta = \frac{2}{5}$

1.4.26 $\cos\ \theta = 1$

For Exercises 27-31, find the exact values of $\cos\ \theta$ and $\tan\ \theta$ when $\sin\ \theta$ has the indicated value.

1.4.27 $\sin\ \theta = \frac{1}{2}$

1.4.28 $\sin\ \theta = -\frac{1}{2}$

1.4.29 $\sin\ \theta = 0$

1.4.30 $\sin\ \theta = -\frac{2}{3}$

1.4.31 $\sin\ \theta = 1$

For Exercises 32-36, find the exact values of $\sin\ \theta$ and $\cos\ \theta$ when $\tan\ \theta$ has the indicated value.

1.4.32 $\tan\ \theta = \frac{1}{2}$

1.4.33 $\tan\ \theta = -\frac{1}{2}$

1.4.34 $\tan\ \theta = 0$

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1.4.35 $\tan\ \theta = \frac{5}{12}$

1.4.36 $\tan\ \theta = 1$

For Exercises 37-40, use Table 1.3 to answer the following questions.

1.4.37 Does $\sin\ 180^\circ\ +\ \sin\ 45^\circ\ =\ \sin\ 225^\circ\ ?$

1.4.38 Does $\tan\ 300^\circ\ -\ \tan\ 30^\circ\ =\ \tan\ 270^\circ\ ?$

1.4.39 Does $\cos\ 180^\circ\ -\ \cos\ 60^\circ\ =\ \cos\ 120^\circ\ ?$

1.4.40 Does $\cos\ 240^\circ\ =\ (\cos\ 120^\circ)^2\ -\ (\sin\ 120^\circ)^2\ ?$

1.4.41 Expand Table 1.3 to include all integer multiples of 15°. See Example 1.10 in Section 1.2.

1.5 Exercises

1.5.1 Let $\theta = 32^\circ$. Find the angle between 0° and 360° which is the

(a) reflection of θ around the x-axis
(b) reflection of θ around the y-axis
(c) reflection of θ around the origin

1.5.2 Repeat Exercise 1 with $\theta = 248^\circ$.

1.5.3 Repeat Exercise 1 with $\theta = -248^\circ$.

1.5.4 We proved Equations 1.4-1.6 for any angle θ in QI. Mimic that proof to show that the formulas hold for θ in QII.

1.5.5 Verify Equations 1.4-1.6 for θ on the coordinate axes, i.e. for $\theta = 0^\circ$, 90°, 180°, 270°.

1.5.6 In Example 1.26 we used the formulas involving $\theta + 90^\circ$ to prove that the slopes of perpendicular lines are negative reciprocals. Show that this result can also be proved using the formulas involving $\theta - 90^\circ$. (*Hint: Only the last paragraph in that example needs to be modified.*)

For Exercises 7-14, find all angles $0^\circ \leq \theta < 360^\circ$ which satisfy the given equation:

1.5.7 $\sin\ \theta = 0.4226$

1.5.8 $\sin\ \theta = 0.1909$

1.5.9 $\cos\ \theta = 0.4226$

1.5.10 $\sin\ \theta = 0$

1.5.11 $\tan\ \theta = 0.7813$

1.5.12 $\sin\ \theta = -0.6294$

1.5.13 $\cos\ \theta = -0.9816$

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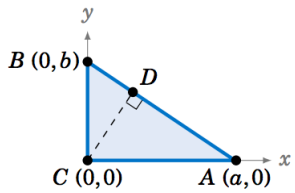
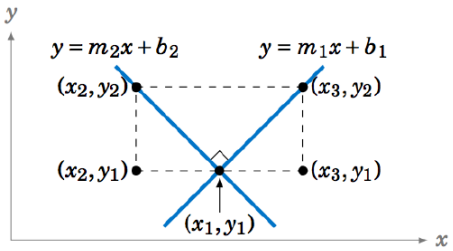
1.5.14 $\tan\theta = -9.514$

1.5.15 In our proof of the Pythagorean Theorem in Section 1.2, we claimed that in a right triangle $\triangle ABC$ it was possible to draw a line segment $\overline{CD}$ from the right angle vertex C to a point D on the hypotenuse $\overline{AB}$ such that $\overline{CD} \perp \overline{AB}$. Use the picture below to prove that claim. (*Hint: Notice how $\triangle ABC$ is placed on the xy-coordinate plane. What is the slope of the hypotenuse? What would be the slope of a line perpendicular to it?*) Also, find the (x, y) coordinates of the point D in terms of a and b.



1.5.16 It can be proved without using trigonometric functions that the slopes of perpendicular lines are negative reciprocals. Let $y = m_1x + b_1$ and $y = m_2x + b_2$ be perpendicular lines (with nonzero slopes), as in the picture below. Use the picture to show that $m_2 = -\frac{1}{m_1}$. (*Hint: Think of similar triangles and the definition of slope.*)



1.5.17 Prove Equations 1.19-1.21 by using Equations 1.10-1.12 and 1.13-1.15.

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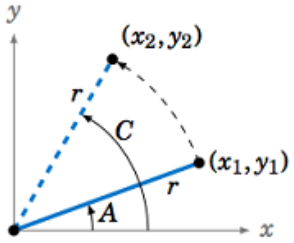
CHAPTER OVERVIEW

2: GENERAL TRIANGLES

Though the methods described will work for right triangles, they are mostly used to solve oblique triangles, that is, triangles which do not have a right angle. There are two types of oblique triangles: an acute triangle has all acute angles, and an obtuse triangle has one obtuse angle. As we will see, Cases 1 and 2 can be solved using the law of sines, Case 3 can be solved using either the law of cosines or the law of tangents, and Case 4 can be solved using the law of cosines.



2.1: THE LAW OF SINES

The Law of Sines states that the sides of a triangle are proportional to the sines of their opposite angles.

2.2: THE LAW OF COSINES

We will now discuss how to solve a triangle where two sides and the angle between them are known. We will use the Law of Cosines to solve this problem.

2.3: THE LAW OF TANGENTS

Law of Tangents is an alternative to the Law of Cosines for Case 3 scenarios (two sides and the included angle). Related to the Law of Tangents are Mollweide's equations.

2.4: THE AREA OF A TRIANGLE

In elementary geometry you learned that the area of a triangle is one-half the base times the height. We will now use that, combined with some trigonometry, to derive more formulas for the area when given various parts of the triangle.

2.5: CIRCUMSCRIBED AND INSCRIBED CIRCLES

Recall from the Law of Sines that any triangle has a common ratio of sides to sines of opposite angles. This common ratio has a geometric meaning: it is the diameter (i.e. twice the radius) of the unique circle in which the trianble can be inscribed, called the circumscribed circle of the triangle.

2.E: GENERAL TRIANGLES (EXERCISES)

These are homework exercises to accompany Corral's "Elementary Trigonometry" Textmap. This is a text on elementary trigonometry, designed for students who have completed courses in high-school algebra and geometry. Though designed for college students, it could also be used in high schools. The traditional topics are covered, but a more geometrical approach is taken than usual. Also, some numerical methods (e.g. the secant method for solving trigonometric equations) are discussed.

2.1: The Law of Sines

Theorem 2.1.1: Law of Sines

If a triangle has sides of lengths a, b, and c opposite the angles A, B, and C, respectively, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\,. \tag{2.1.1}$$

Note that by taking reciprocals, Equation 2.1.1 can be written as

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}\,, \tag{2.1.2}$$

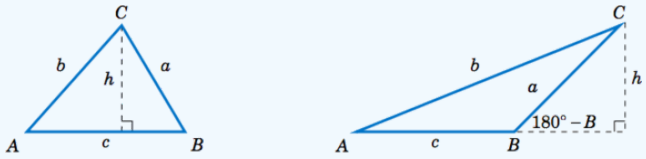
and it can also be written as a collection of three equations:

$$\frac{a}{b} = \frac{\sin A}{\sin B}\,, \quad \frac{a}{c} = \frac{\sin A}{\sin C}\,, \quad \frac{b}{c} = \frac{\sin B}{\sin C} \tag{2.1.3}$$

Another way of stating the Law of Sines is: *The sides of a triangle are proportional to the sines of their opposite angles.*

Proof

To prove the Law of Sines, let $\triangle ABC$ be an oblique triangle. Then $\angle ABC$ can be acute, as in Figure 2.1.1(a), or it can be obtuse, as in Figure 2.1.1(b). In each case, draw the *altitude* from the vertex at C to the side $\overline{AB}$. In Figure 2.1.1(a) the altitude lies inside the triangle, while in Figure 2.1.1(b) the altitude lies outside the triangle.



(a) Acute triangle (b) Obtuse triangle

Figure 2.1.1: Proof of the Law of Sines for an oblique triangle $\triangle ABC$

Let h be the height of the altitude. For each triangle in Figure 2.1.1, we see that

$$\frac{h}{b} = \sin A \tag{2.1.4}$$

and

$$\frac{h}{a} = \sin B \tag{2.1.5}$$

in Figure 2.1.1(b), $\frac{h}{a} = \sin(180^\circ - B) = \sin B$ by Equation (1.19) in Section 1.5). Thus, solving for h in Equation 2.1.5 and substituting that into Equation 2.1.4 gives

$$\frac{a \sin B}{b} = \sin A\,, \tag{2.1.6}$$

and so putting a and A on the left side and b and B on the right side, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B}\,. \tag{2.1.7}$$

By a similar argument, drawing the altitude from A to $\overline{BC}$ gives

$$\frac{b}{\sin B} = \frac{c}{\sin C}\,, \tag{2.1.8}$$

so putting the last two equations together proves the theorem.

□

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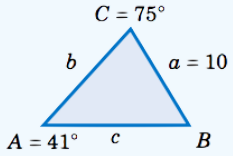
Note that we did not prove the Law of Sines for right triangles, since it turns out (see Exercise 12) to be trivially true for that case.

Example 2.1.1: Case 1 - One side and two angles known

Solve the triangle $\triangle ABC$ given $a = 10$, $A = 41^\circ$, and $C = 75^\circ$.



Solution

We can find the third angle by subtracting the other two angles from 180°, then use the law of sines to find the two unknown sides. In this example we need to find B, b, and c. First, we see that

$$B = 180^\circ - A - C = 180^\circ - 41^\circ - 75^\circ \quad \Rightarrow \quad \boxed{B = 64^\circ}\,.$$

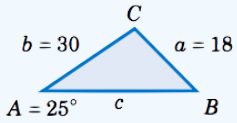
So by the Law of Sines we have

$$\frac{b}{\sin B} = \frac{a}{\sin A} \quad \Rightarrow \quad b = \frac{a \sin B}{\sin A} = \frac{10 \sin 64^\circ}{\sin 41^\circ} \quad \Rightarrow \quad \boxed{b = 13.7}\text{, and}$$

$$\frac{c}{\sin C} = \frac{a}{\sin A} \quad \Rightarrow \quad c = \frac{a \sin C}{\sin A} = \frac{10 \sin 75^\circ}{\sin 41^\circ} \quad \Rightarrow \quad \boxed{c = 14.7}\,.$$

Example 2.1.2: Case 2 - Two sides and one opposite angle known

Solve the triangle $\triangle ABC$ given $a = 18$, $A = 25^\circ$, and $b = 30$.



Solution

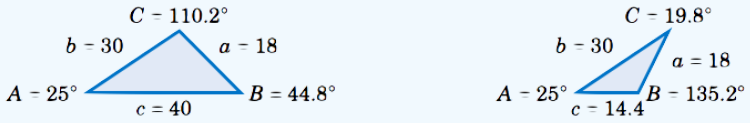
In this example we know the side a and its opposite angle A, and we know the side b. We can use the Law of Sines to find the other opposite angle B, then find the third angle C by subtracting A and B from 180°, then use the law of sines to find the third side c. By the Law of Sines, we have

$$\frac{\sin B}{b} = \frac{\sin A}{a} \quad \Rightarrow \quad \sin B = \frac{b \sin A}{a} = \frac{30 \sin 25^\circ}{18} \quad \Rightarrow \quad \sin B = 0.7044\,.$$

Using the $\boxed{\sin^{-1}}$ button on a calculator gives $B = 44.8^\circ$. However, recall from Section 1.5 that $\sin(180^\circ - B) = \sin B$. So there is a second possible solution for B, namely $180^\circ - 44.8^\circ = 135.2^\circ$. Thus, we have to solve *twice* for C and c: once for $B = 44.8^\circ$ and once for $B = 135.2^\circ$:

$$\begin{array}{c|c} \boxed{B = 44.8^\circ} & \boxed{B = 135.2^\circ} \\ C = 180^\circ - A - B = 180^\circ - 25^\circ - 44.8^\circ = 110.2^\circ & C = 180^\circ - A - B = 180^\circ - 25^\circ - 135.2^\circ = 19.8^\circ \\ \dfrac{c}{\sin C} = \dfrac{a}{\sin A} \Rightarrow c = \dfrac{a \sin C}{\sin A} = \dfrac{18 \sin 110.2^\circ}{\sin 25^\circ} & \dfrac{c}{\sin C} = \dfrac{a}{\sin A} \Rightarrow c = \dfrac{a \sin C}{\sin A} = \dfrac{18 \sin 19.8^\circ}{\sin 25^\circ} \\ \Rightarrow c = 40 & \Rightarrow c = 14.4 \end{array} \qquad (2.1.9)$$

Hence, $\boxed{B = 44.8^\circ, C = 110.2^\circ, c = 40}$ and $\boxed{B = 135.2^\circ, C = 19.8^\circ, c = 14.4}$ are the two possible sets of solutions. This means that there are two possible triangles, as shown in Figure 2.1.2.



(a) $B = 44.8°$ (b) $B = 135.2°$

Figure 2.1.2: Two possible solutions

In Example 2.1.2 we saw what is known as the *ambiguous case*. That is, there may be more than one solution. It is also possible for there to be exactly one solution or no solution at all.

Example 2.1.3: Case 2 - Two sides and one opposite angle known

Solve the triangle $\triangle ABC$ given $a = 5$, $A = 30°$, and $b = 12$.

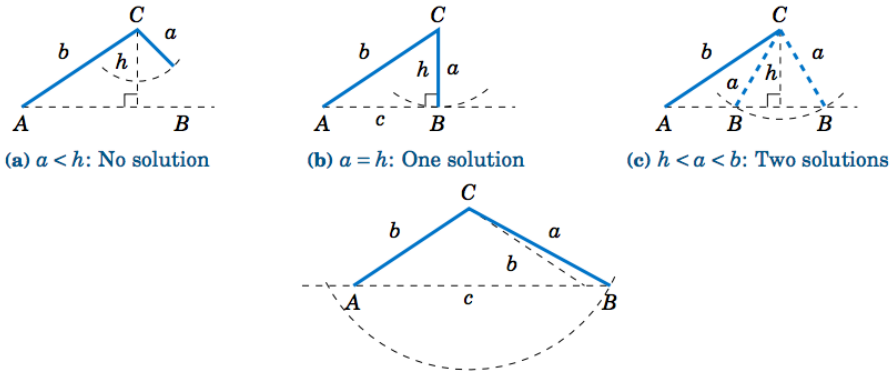
Solution

By the Law of Sines, we have

$$\frac{\sin B}{b} = \frac{\sin A}{a} \quad \Rightarrow \quad \sin B = \frac{b \sin A}{a} = \frac{12 \sin 30°}{5} \quad \Rightarrow \quad \sin B = 1.2\,,$$

which is impossible since $|\sin B| \leq 1$ for any angle B. Thus, there is $\boxed{\text{no solution}}$.

There is a way to determine how many solutions a triangle has in Case 2. For a triangle $\triangle ABC$, suppose that we know the sides a and b and the angle A. Draw the angle A and the side b, and imagine that the side a is attached at the vertex at C so that it can "swing" freely, as indicated by the dashed arc in Figure 2.1.3 below.



(a) $a < h$: No solution (b) $a = h$: One solution (c) $h < a < b$: Two solutions

(d) $a \geq b$: One solution

Figure 2.1.3: The ambiguous case when A is acute

If A is acute, then the altitude from C to $\overline{AB}$ has height $h = b \sin A$. As we can see in Figure 2.1.3(a)-(c), there is no solution when $a < h$ (this was the case in Example 2.3); there is exactly one solution - namely, a right triangle - when $a = h$; and there are two solutions when $h < a < b$ (as was the case in Example 2.2). When $a \geq b$ there is only one solution, even though it appears from Figure 2.1.3(d) that there may be two solutions, since the dashed arc intersects the horizontal line at two points. However, the point of intersection to the left of A in Figure 2.1.3(d) can not be used to determine B, since that would make A an obtuse angle, and we assumed that A was acute.

If A is not acute (i.e. A is obtuse or a right angle), then the situation is simpler: there is no solution if $a \leq b$, and there is exactly one solution if $a > b$ (see Figure 2.1.4).

(a) $a \leq b$: No solution (b) $a > b$: One solution

Figure 2.1.4: The ambiguous case when $A \geq 90°$

Table 2.1 summarizes the ambiguous case of solving $\triangle ABC$ when given a, A, and b. Of course, the letters can be interchanged, e.g. replace a and A by c and C, etc.

Table 2.1 Summary of the ambiguous case

$0° < A < 90°$		$90° \le A < 180°$	
$a < b \sin A$:	No solution	$a \le b$:	No solution
$a = b \sin A$:	One solution	$a > b$:	One solution
$b \sin A < a < b$:	Two solutions		
$a \ge b$:	One solution		

There is an interesting geometric consequence of the Law of Sines. Recall from Section 1.1 that in a right triangle the hypotenuse is the largest side. Since a right angle is the largest angle in a right triangle, this means that the largest side is opposite the largest angle. What the Law of Sines does is generalize this to *any* triangle:

In any triangle, the largest side is opposite the largest angle.

Proof

To prove this, let C be the largest angle in a triangle $\triangle ABC$. If $C = 90°$ then we already know that its opposite side c is the largest side. So we just need to prove the result for when C is acute and for when C is obtuse. In both cases, we have $A \le C$ and $B \le C$. We will first show that $\sin A \le \sin C$ and $\sin B \le \sin C$.

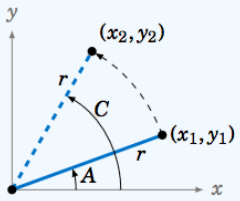


Figure 2.1.5

If C is acute, then A and B are also acute. Since $A \le C$, imagine that A is in standard position in the xy-coordinate plane and that we rotate the terminal side of A counterclockwise to the terminal side of the larger angle C, as in Figure 2.1.5. If we pick points (x_1, y_1) and (x_2, y_2) on the terminal sides of A and C, respectively, so that their distance to the origin is the same number r, then we see from the picture that $y_1 \le y_2$, and hence

$$\sin A = \frac{y_1}{r} \le \frac{y_2}{r} = \sin C .$$

By a similar argument, $B \le C$ implies that $\sin B \le \sin C$. Thus, $\sin A \le \sin C$ and $\sin B \le \sin C$ when C is acute. We will now show that these inequalities hold when C is obtuse.

If C is obtuse, then $180° - C$ is acute, as are A and B. If $A > 180° - C$ then $A + C > 180°$, which is impossible. Thus, we must have $A \le 180° - C$. Likewise, $B \le 180° - C$. So by what we showed above for acute angles, we know that $\sin A \le \sin (180° - C)$ and $\sin B \le \sin (180° - C)$. But we know from Section 1.5 that $\sin C = \sin (180° - C)$. Hence, $\sin A \le \sin C$ and $\sin B \le \sin C$ when C is obtuse.

Thus, $\sin A \le \sin C$ if C is acute or obtuse, so by the Law of Sines we have

$$\frac{a}{c} = \frac{\sin A}{\sin C} \le \frac{\sin C}{\sin C} = 1 \quad \Rightarrow \quad \frac{a}{c} \le 1 \quad \Rightarrow \quad a \le c .$$

By a similar argument, $b \le c$. Thus, $a \le c$ and $b \le c$, i.e. c is the largest side.

□

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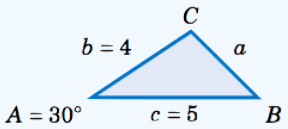
2.2: The Law of Cosines

We will now discuss how to solve a triangle in Case 3: two sides and the angle between them. First, let us see what happens when we try to use the Law of Sines for this case.

Example 2.2.1: Case 3 - Two sides and the angle between them Known

Solve the triangle $\triangle ABC$ given $A = 30^\circ$, $b = 4$, and $c = 5$.



Solution

Using the Law of Sines, we have

$$\frac{a}{\sin\ 30^\circ} = \frac{4}{\sin\ B} = \frac{5}{\sin\ C},$$

where each of the equations has two unknown parts, making the problem impossible to solve. For example, to solve for a we could use the equation

$$\frac{4}{\sin\ B} = \frac{5}{\sin\ C}$$

to solve for $\sin\ B$ in terms of $\sin\ C$ and substitute that into the equation

$$\frac{a}{\sin\ 30^\circ} = \frac{4}{\sin\ B}.$$

But that would just result in the equation

$$\frac{a}{\sin\ 30^\circ} = \frac{5}{\sin\ C},$$

which we already knew and which still has two unknowns! Thus, this problem can not be solved using the Law of Sines.

To solve the triangle in the above example, we can use the *Law of Cosines:*

Theorem 2.2.1: Law of Cosines

If a triangle has sides of lengths a, b, and c opposite the angles A, B, and C, respectively, then

$$a^2 = b^2 + c^2 - 2bc\ \cos\ A, \tag{2.2.1}$$

$$b^2 = c^2 + a^2 - 2ca\ \cos\ B, \tag{2.2.2}$$

$$c^2 = a^2 + b^2 - 2ab\ \cos\ C. \tag{2.2.3}$$

Proof

To prove the Law of Cosines, let $\triangle ABC$ be an oblique triangle. Then $\triangle ABC$ can be acute, as in Figure 2.2.1*a*, or it can be obtuse, as in Figure 2.2.1*b*. In each case, draw the altitude from the vertex at C to the side $\overline{AB}$. In Figure 2.2.1*a*, the altitude divides $\overline{AB}$ into two line segments with lengths x and $c - x$, while in Figure 2.2.1*b* the altitude extends the side $\overline{AB}$ by a distance x. Let h be the height of the altitude.

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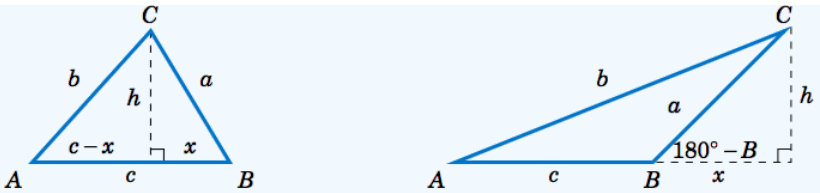


(a) Acute triangle (b) Obtuse triangle

Figure 2.2.1: Proof of the Law of Cosines for an oblique triangle $\triangle ABC$

For each triangle in Figure 2.2.1, we see by the Pythagorean Theorem that

$$h^2 = a^2 - x^2 \tag{2.2.4}$$

and likewise for the acute triangle in Figure 2.2.1a we see that

$$b^2 = h^2 + (c-x)^2 . \tag{2.2.5}$$

Thus, substituting the expression for h^2 in Equation 2.2.4 into Equation 2.2.5 gives

$$\begin{aligned} b^2 &= a^2 - x^2 + (c-x)^2 \\ &= a^2 - x^2 + c^2 - 2cx + x^2 \\ &= a^2 + c^2 - 2cx . \end{aligned}$$

But we see from Figure 2.2.1a that $x = a\ \cos\ B$, so

$$b^2 = a^2 + c^2 - 2ca\ \cos\ B . \tag{2.2.6}$$

And for the obtuse triangle in Figure 2.2.1b we see that

$$b^2 = h^2 + (c+x)^2 . \tag{2.2.7}$$

Thus, substituting the expression for h^2 in Equation 2.2.4 into Equation 2.2.7 gives

$$\begin{aligned} b^2 &= a^2 - x^2 + (c+x)^2 \\ &= a^2 - x^2 + c^2 + 2cx + x^2 \\ &= a^2 + c^2 + 2cx . \end{aligned}$$

But we see from Figure 2.2.1a that $x = a\ \cos\ (180^\circ - B)$, and we know from Section 1.5 that $\cos\ (180^\circ - B) = -\cos\ B$. Thus, $x = -a\ \cos\ B$ and so

$$b^2 = a^2 + c^2 - 2ca\ \cos\ B . \tag{2.2.8}$$

So for both acute and obtuse triangles we have proved Equation 2.2.2 in the Law of Cosines. Notice that the proof was for B acute and obtuse. By similar arguments for A and C we get the other two formulas.

□

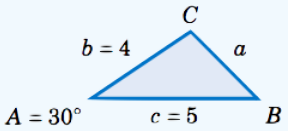
Note that we did not prove the Law of Cosines for right triangles, since it turns out (see Exercise 15) that all three formulas reduce to the Pythagorean Theorem for that case. The Law of Cosines can be viewed as a generalization of the Pythagorean Theorem.

Also, notice that it suffices to remember just one of the three Equations 2.2.1-2.2.3, since the other two can be obtained by "cycling" through the letters a, b, and c. That is, replace a by b, replace b by c, and replace c by a (likewise for the capital letters). One cycle will give you the second formula, and another cycle will give you the third.

The angle between two sides of a triangle is often called the **included angle**. Notice in the Law of Cosines that if two sides and their included angle are known (e.g. b, c, and A), then we have a formula for the square of the third side. We will now solve the triangle from Example 2.2.2.

Example 2.2.2: Case 3 - Two sides and the angle between them Known

Solve the triangle $\triangle ABC$ given $A = 30^\circ$, $b = 4$, and $c = 5$.



Solution

We will use the Law of Cosines to find a, use it again to find B, then use $C = 180^\circ - A - B$. First, we have

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 4^2 + 5^2 - 2(4)(5)\cos 30^\circ = 6.36 \quad \Rightarrow \quad \boxed{a = 2.52}. \end{aligned}$$

Now we use the formula for b^2 to find B:

$$\begin{aligned} b^2 = c^2 + a^2 - 2ca\cos B \quad &\Rightarrow \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca} \\ &\Rightarrow \quad \cos B = \frac{5^2 + (2.52)^2 - 4^2}{2(5)(2.52)} = 0.6091 \\ &\Rightarrow \quad \boxed{B = 52.5^\circ} \end{aligned}$$

Thus, $C = 180^\circ - A - B = 180^\circ - 30^\circ - 52.5^\circ \Rightarrow \boxed{C = 97.5^\circ}$.

Notice in Example 2.2.2 that there was only one solution. For Case 3 this will *always* be true: when given two sides and their included angle, the triangle will have exactly one solution. The reason is simple: when joining two line segments at a common vertex to form an angle, there is exactly one way to connect their free endpoints with a third line segment, regardless of the size of the angle.

You may be wondering why we used the Law of Cosines a second time in Example 2.2.2, to find the angle B. Why not use the Law of Sines, which has a simpler formula? The reason is that using the cosine function eliminates any ambiguity: if the cosine is positive then the angle is acute, and if the cosine is negative then the angle is obtuse. This is in contrast to using the sine function; as we saw in Section 2.1, both an acute angle and its obtuse supplement have the same positive sine.

To see this, suppose that we had used the Law of Sines to find B in Example 2.2.2:

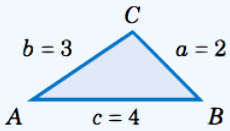
$$\sin B = \frac{b \sin A}{a} = \frac{4 \sin 30^\circ}{2.52} = 0.7937 \quad \Rightarrow \quad B = 52.5^\circ \text{ or } 127.5^\circ$$

How would we know which answer is correct? We could not immediately rule out $B = 127.5^\circ$ as too large, since it would make $A + B = 157.5^\circ < 180^\circ$ and so $C = 22.5^\circ$, which seems like it could be a valid solution. However, this solution is impossible. Why? Because the largest side in the triangle is $c = 5$, which (as we learned in Section 2.1) means that C has to be the largest angle. But $C = 22.5^\circ$ would not be the largest angle in this solution, and hence we have a contradiction.

It remains to solve a triangle in Case 4, i.e. given three sides. We will now see how to use the Law of Cosines for that case.

Example 2.2.3: Case 4 - Three sides Known

Solve the triangle $\triangle ABC$ given $a = 2$, $b = 3$, and $c = 4$.



Solution

We will use the Law of Cosines to find B and C, then use $A = 180^\circ - B - C$. First, we use the formula for b^2 to find B:

$$b^2 = c^2 + a^2 - 2ca\cos B \quad \Rightarrow \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}$$
$$\Rightarrow \quad \cos B = \frac{4^2 + 2^2 - 3^2}{2(4)(2)} = 0.6875$$
$$\Rightarrow \quad \boxed{B = 46.6^\circ}$$

Now we use the formula for c^2 to find C:

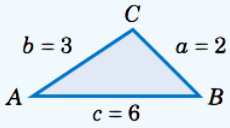
$$c^2 = a^2 + b^2 - 2ab\cos C \quad \Rightarrow \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\Rightarrow \quad \cos C = \frac{2^2 + 3^2 - 4^2}{2(2)(3)} = -0.25$$
$$\Rightarrow \quad \boxed{C = 104.5^\circ}$$

Thus, $A = 180^\circ - B - C = 180^\circ - 46.6^\circ - 104.5^\circ \Rightarrow \boxed{A = 28.9^\circ}$.

It may seem that there is always a solution in Case 4 (given all three sides), but that is not true, as the following example shows.

Example 2.2.4: Case 4 - Three sides Known

Solve the triangle $\triangle ABC$ given $a = 2$, $b = 3$, and $c = 6$.



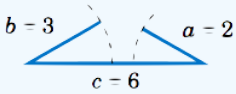
Solution

If we blindly try to use the Law of Cosines to find A, we get

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \Rightarrow \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{3^2 + 6^2 - 2^2}{2(3)(6)} = 1.139\,,$$

which is impossible since $|\cos A| \le 1$. Thus, there is $\boxed{\text{no solution}}$.

We could have saved ourselves some effort by recognizing that the length of one of the sides ($c = 6$) is greater than the sums of the lengths of the remaining sides ($a = 2$ and $b = 3$), which (as the picture below shows) is impossible in a triangle.



The Law of Cosines can also be used to solve triangles in Case 2 (two sides and one opposite angle), though it is less commonly used for that purpose than the Law of Sines. The following example gives an idea of how to do this.

Example 2.2.5: Case 2 - Two sides and one opposite angle Known

Solve the triangle $\triangle ABC$ given $a = 18$, $A = 25^\circ$, and $b = 30$.

Solution

In Example 2.2 from Section 2.1 we used the Law of Sines to show that there are two sets of solutions for this triangle: $B = 44.8^\circ$, $C = 110.2^\circ$, $c = 40$ and $B = 135.2^\circ$, $C = 19.8^\circ$, $c = 14.4$. To solve this using the Law of Cosines, first find c by using the formula for a^2:

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \Rightarrow \quad 18^2 = 30^2 + c^2 - 2(30)c\cos 25^\circ$$
$$\Rightarrow \quad c^2 - 54.38\,c + 576 = 0\,,$$

which is a quadratic equation in c, so we know that it can have either zero, one, or two real roots (corresponding to the number of solutions in Case 2). By the quadratic formula, we have

$$c = \frac{54.38 \pm \sqrt{(54.38)^2 - 4(1)(576)}}{2(1)} = 40 \text{ or } 14.4\,.$$

Note that these are the same values for c that we found before. For $c = 40$ we get

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{40^2 + 18^2 - 30^2}{2(40)(18)} = 0.7111 \quad \Rightarrow \quad B = 44.7^\circ \quad \Rightarrow \quad C = 110.3^\circ\,,$$

which is close to what we found before (the small difference being due to different rounding). The other solution set can be obtained similarly.

Like the Law of Sines, the Law of Cosines can be used to prove some geometric facts, as in the following example.

Example 2.2.6: Parallelogram Diagonals

Use the Law of Cosines to prove that the sum of the squares of the diagonals of any parallelogram equals the sum of the squares of the sides.

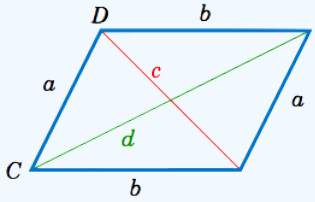


Figure 2.2.2

Solution:

Let a and b be the lengths of the sides, and let the diagonals opposite the angles C and D have lengths c and d, respectively, as in Figure 2.2.2. Then we need to show that

$$c^2 + d^2 = a^2 + b^2 + a^2 + b^2 = 2\left(a^2 + b^2\right).$$

By the Law of Cosines, we know that

$$c^2 = a^2 + b^2 - 2ab\cos C\,, \text{ and}$$
$$d^2 = a^2 + b^2 - 2ab\cos D\,.$$

By properties of parallelograms, we know that $D = 180^\circ - C$, so

$$d^2 = a^2 + b^2 - 2ab\cos(180^\circ - C)$$
$$= a^2 + b^2 + 2ab\cos C\,,$$

since $\cos(180^\circ - C) = -\cos C$. Thus,

$$c^2 + d^2 = a^2 + b^2 - 2ab\cos C + a^2 + b^2 + 2ab\cos C$$
$$= 2\left(a^2 + b^2\right).$$

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2.3: The Law of Tangents

We have shown how to solve a triangle in all four cases discussed at the beginning of this chapter. An alternative to the Law of Cosines for Case 3 (two sides and the included angle) is the *Law of Tangents*:

If a triangle has sides of lengths a, b, and c opposite the angles A, B, and C, respectively, then

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}, \tag{2.3.1}$$

$$\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}, \tag{2.3.2}$$

$$\frac{c-a}{c+a} = \frac{\tan\frac{1}{2}(C-A)}{\tan\frac{1}{2}(C+A)}. \tag{2.3.3}$$

Note that since $\tan(-\theta) = -\tan\theta$ for any angle θ, we can switch the order of the letters in each of the above formulas. For example, we can rewrite Equation 2.3.1 as

$$\frac{b-a}{b+a} = \frac{\tan\frac{1}{2}(B-A)}{\tan\frac{1}{2}(B+A)}, \tag{2.3.4}$$

and similarly for the other formulas. If $a > b$, then it is usually more convenient to use Equation 2.3.1, while Equation 2.3.4 is more convenient when $b > a$.

$A+B+C = 180^\circ$, so $A+B = 180^\circ - C = 180^\circ - 96^\circ = 84^\circ$. Thus, by the Law of Tangents,

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} \quad\Rightarrow\quad \frac{5-3}{5+3} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(84^\circ)}$$
$$\Rightarrow\quad \tan\tfrac{1}{2}(A-B) = \tfrac{2}{8}\tan 42^\circ = 0.2251$$
$$\Rightarrow\quad \tfrac{1}{2}(A-B) = 12.7^\circ \quad\Rightarrow\quad A-B = 25.4^\circ.$$

We now have two equations involving A and B, which we can solve by adding the equations:

$$\begin{array}{rl} A-B &= 25.4^\circ \\ A+B &= 84^\circ \\ \hline 2A &= 109.4^\circ \end{array} \quad\Rightarrow\quad \boxed{A = 54.7^\circ} \quad\Rightarrow\quad B = 84^\circ - 54.7^\circ \quad\Rightarrow\quad \boxed{B = 29.3^\circ}$$

We can find the remaining side c by using the Law of Sines:

$$c = \frac{a\sin C}{\sin A} = \frac{5\sin 96^\circ}{\sin 54.7^\circ} \quad\Rightarrow\quad \boxed{c = 6.09}$$

Note that in any triangle $\triangle ABC$, if $a = b$ then $A = B$ (why?), and so both sides of Equation 2.3.1 would be 0 (since $\tan 0^\circ = 0$). This means that *the Law of Tangents is of no help in Case 3 when the two known sides are equal*. For this reason, and perhaps also because of the somewhat unusual way in which it is used, the Law of Tangents seems to have fallen out of favor in trigonometry books lately. It does not seem to have any advantages over the Law of Cosines, which works even when the sides are equal, requires slightly fewer steps, and is perhaps more straightforward.

Related to the Law of Tangents are *Mollweide's equations*:

Mollweide's equations:

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For any triangle $\triangle ABC$,

$$\frac{a-b}{c} = \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}, \text{ and} \tag{2.3.5}$$

$$\frac{a+b}{c} = \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}. \tag{2.3.6}$$

Note that all six parts of a triangle appear in both of Mollweide's equations. For this reason, either equation can be used to check a solution of a triangle. If both sides of the equation agree (more or less), then we know that the solution is correct.

Solution

Recall that the full solution was $a=5$, $b=3$, $c=6.09$, $A=54.7^\circ$, $B=29.3^\circ$, and $C=96^\circ$. We will check this with Equation 2.3.5:

$$\begin{aligned}
\frac{a-b}{c} &= \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C} \\
\frac{5-3}{6.09} &= \frac{\sin\frac{1}{2}(54.7^\circ-29.3^\circ)}{\cos\frac{1}{2}(96^\circ)} \\
\frac{2}{6.09} &= \frac{\sin 12.7^\circ}{\cos 48^\circ} \\
0.3284 &= 0.3285 \quad \checkmark
\end{aligned}$$

The small difference (≈ 0.0001) is due to rounding errors from the original solution, so we can conclude that both sides of the equation agree, and hence the solution is correct.

Example 2.12

Can a triangle have the parts $a=6$, $b=7$, $c=9$, $A=55^\circ$, $B=60^\circ$, and $C=65^\circ$?

Solution:

Before using Mollweide's equations, simpler checks are that the angles add up to 180° and that the smallest and largest sides are opposite the smallest and largest angles, respectively. In this case all those conditions hold. So check with Mollweide's Equation 2.3.6:

$$\begin{aligned}
\frac{a+b}{c} &= \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C} \\
\frac{6+7}{9} &= \frac{\cos\frac{1}{2}(55^\circ-60^\circ)}{\sin\frac{1}{2}(65^\circ)} \\
\frac{13}{9} &= \frac{\cos(-2.5^\circ)}{\sin 32.5^\circ} \\
1.44 &= 1.86 \quad \times
\end{aligned}$$

Here the difference is far too large, so we conclude that there is no triangle with these parts.

We will prove the Law of Tangents and Mollweide's equations in Chapter 3, where we will be able to supply brief analytic proofs.

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2.4: The Area of a Triangle

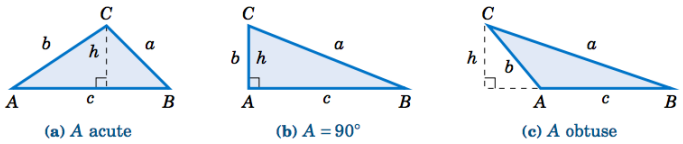


Figure 2.4.1 Area of $\triangle ABC$

In each case we draw an altitude of height h from the vertex at C to $\overline{AB}$, so that the area (which we will denote by the letter K) is given by $K = \frac{1}{2}hc$. But we see that $h = b \sin A$ in each of the triangles (since $h = b$ and $\sin A = \sin 90^\circ = 1$ in Figure 2.4.1(b), and $h = b \sin (180^\circ - A) = b \sin A$ in Figure 2.4.1(c)). We thus get the following formula:

$$\boxed{\text{Area} = K = \tfrac{1}{2}bc \sin A} \tag{2.4.1}$$

The above formula for the area of $\triangle ABC$ is in terms of the known parts A, b, and c. Similar arguments for the angles B and C give us:

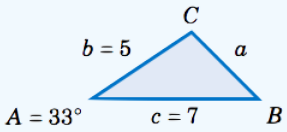
$$\text{Area} = K = \tfrac{1}{2}ac \sin B \tag{2.4.2}$$

$$\text{Area} = K = \tfrac{1}{2}ab \sin C \tag{2.4.3}$$

Notice that the height h does not appear explicitly in these formulas, although it is implicitly there. These formulas have the advantage of being in terms of parts of the triangle, without having to find h separately.

Example 2.13

Find the area of the triangle $\triangle ABC$ given $A = 33^\circ$, $b = 5$, and $c = 7$.



Solution:

Using Equation 2.4.1, the area K is given by:

$$\begin{aligned} K &= \tfrac{1}{2}bc \sin A \\ &= \tfrac{1}{2}(5)(7) \sin 33^\circ \\ K &= 9.53 \end{aligned}$$

Case 2: Three angles and any side.

Suppose that we have a triangle $\triangle ABC$ in which one side, say, a, and all three angles are known. By the Law of Sines we know that

$$c = \frac{a \sin C}{\sin A},$$

so substituting this into Equation 2.4.2 we get:

$$\boxed{\text{Area} = K = \frac{a^2 \sin B \sin C}{2 \sin A}} \tag{2.4.4}$$

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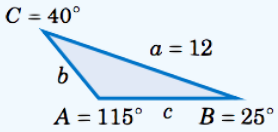
Similar arguments for the sides b and c give us:

$$\text{Area} = K = \frac{b^2 \sin A \sin C}{2 \sin B} \tag{2.4.5}$$

$$\text{Area} = K = \frac{c^2 \sin A \sin B}{2 \sin C} \tag{2.4.6}$$

Find the area of the triangle $\triangle ABC$ given\ $A = 115°$, $B = 25°$, $C = 40°$, and $a = 12$.



Solution:

Using Equation 2.4.4, the area K is given by:

$$\begin{aligned} K &= \frac{a^2 \sin B \sin C}{2 \sin A} \\ &= \frac{12^2 \sin 25° \sin 40°}{2 \sin 115°} \\ K &= 21.58 \end{aligned}$$

Case 3: Three sides.

Suppose that we have a triangle $\triangle ABC$ in which all three sides are known. Then *Heron's formula* gives us the area:

Heron's formula

For a triangle $\triangle ABC$ with sides a, b, and c, let $s = \frac{1}{2}(a+b+c)$ (i.e. $2s = a+b+c$ is the perimeter of the triangle). Then the area K of the triangle is

$$\text{Area} = K = \sqrt{s(s-a)(s-b)(s-c)} \ . \tag{2.4.7}$$

To prove this, first remember that the area K is one-half the base times the height. Using c as the base and the altitude h as the height, as before in Figure 2.4.1, we have $K = \frac{1}{2}hc$. Squaring both sides gives us

$$K^2 = \tfrac{1}{4} h^2 c^2 \ . \tag{2.4.8}$$

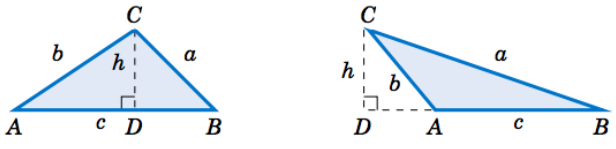


Figure 2.4.2 Proof of Heron's formula

By the Pythagorean Theorem, we see that $h^2 = b^2 - (AD)^2$. In Figure 2.4.2(a), we see that $AD = b \cos A$. And in Figure 2.4.2(b) we see that $AD = b \cos(180° - A) = -b \cos A$. Hence, in either case we have $(AD)^2 = b^2 (\cos A)^2$, and so

$$h^2 = b^2 - b^2 (\cos A)^2 = b^2 (1 - (\cos A)^2) = b^2 (1 + \cos A)(1 - \cos A) \, . \tag{2.4.9}$$

(Note that the above equation also holds when $A = 90°$ since $\cos 90° = 0$ and $h = b$). Thus, substituting Equation 2.4.9 into Equation 2.4.8, we have

$$K^2 = \tfrac{1}{4}\, b^2 c^2 \,(1+\cos\ A)\,(1-\cos\ A)\,. \tag{2.4.10}$$

By the Law of Cosines we know that

$$\begin{aligned} 1+\cos\ A &= 1+\frac{b^2+c^2-a^2}{2bc} = \frac{2bc+b^2+c^2-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc} = \frac{((b+c)+a)\,((b+c)-a)}{2bc} \\ &= \frac{(a+b+c)\,(b+c-a)}{2bc}\,, \end{aligned}$$

and similarly

$$\begin{aligned} 1-\cos\ A &= 1-\frac{b^2+c^2-a^2}{2bc} = \frac{2bc-b^2-c^2+a^2}{2bc} = \frac{a^2-(b-c)^2}{2bc} = \frac{(a-(b-c))\,(a+(b-c))}{2bc} \\ &= \frac{(a-b+c)\,(a+b-c)}{2bc}\,. \end{aligned}$$

Thus, substituting these expressions into Equation 2.4.10, we have

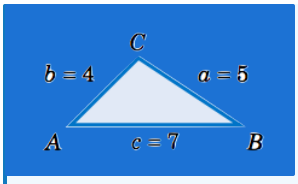
$$\begin{aligned} K^2 &= \tfrac{1}{4}\, b^2 c^2\, \frac{(a+b+c)\,(b+c-a)}{2bc} \cdot \frac{(a-b+c)\,(a+b-c)}{2bc} \\ &= \frac{a+b+c}{2} \cdot \frac{b+c-a}{2} \cdot \frac{a-b+c}{2} \cdot \frac{a+b-c}{2}\,, \end{aligned}$$

and since we defined $s = \frac{1}{2}\,(a+b+c)$, we see that

$$K^2 \;=\; s\,(s-a)\,(s-b)\,(s-c)\,,$$

so upon taking square roots we get

$$K \;=\; \sqrt{s\,(s-a)\,(s-b)\,(s-c)}\quad . \quad \textbf{QED}$$



Solution

Using Heron's formula with $s = \frac{1}{2}\,(a+b+c) = \frac{1}{2}\,(5+4+7) = 8$, the area K is given by:

$$\begin{aligned} K &= \sqrt{s\,(s-a)\,(s-b)\,(s-c)} \\ &= \sqrt{8\,(8-5)\,(8-4)\,(8-7)} \;=\; \sqrt{96} \quad\Rightarrow\quad \boxed{K \;=\; 4\sqrt{6} \;\approx\; 9.8}\,. \end{aligned}$$

Heron's formula is useful for theoretical purposes (e.g. in deriving other formulas). However, it is not well-suited for calculator use, exhibiting what is called *numerical instability* for "extreme" triangles, as in the following example.

Example 2.16

Find the area of the triangle $\triangle ABC$ given $a = 1000000$, $b = 999999.9999979$, and $c = 0.0000029$.

Solution:

To use Heron's formula, we need to calculate $s = \frac{1}{2}\,(a+b+c)$. Notice that the actual value of $a+b+c$ is 2000000.0000008, which has 14 digits. Most calculators can store 12-14 digits internally (even if they display less), and

hence may round off that value of $a+b+c$ to 2000000. When we then divide that rounded value for $a+b+c$ by 2 to get s, some calculators (e.g. the TI-83 Plus) will give a rounded down value of 1000000.

This is a problem because $a=1000000$, and so we would get $s-a=0$, causing Heron's formula to give us an area of 0 for the triangle! And this is indeed the incorrect answer that the TI-83 Plus returns. Other calculators may give some other inaccurate answer, depending on how they store values internally. The actual area - accurate to 15 decimal places - is $K=0.99999999999895$; i.e. it is basically 1.

The above example shows how problematic *floating-point arithmetic* can be. Luckily there is a better formula for the area of a triangle when the three sides are known:

For a triangle $\triangle ABC$ with sides $a \geq b \geq c$, the area is:

$$\text{Area} = K = \frac{1}{4}\sqrt{(a+(b+c))(c-(a-b))(c+(a-b))(a+(b-c))} \tag{2.4.11}$$

To use this formula, sort the names of the sides so that $a \geq b \geq c$. Then perform the operations inside the square root *in the exact order in which they appear in the formula, including the use of parentheses*. Then take the square root and divide by 4. For the triangle in Example 2.16, the above formula gives an answer of exactly $K=1$ on the same TI-83 Plus calculator that failed with Heron's formula. What is amazing about this formula is that it is just Heron's formula rewritten! The use of parentheses is what forces the correct order of operations for numerical stability.

Another formula for the area of a triangle given its three sides is given below:

For a triangle $\triangle ABC$ with sides $a \geq b \geq c$, the area is:

$$\text{Area} = K = \frac{1}{2}\sqrt{a^2c^2 - \left(\frac{a^2+c^2-b^2}{2}\right)^2} \tag{2.4.12}$$

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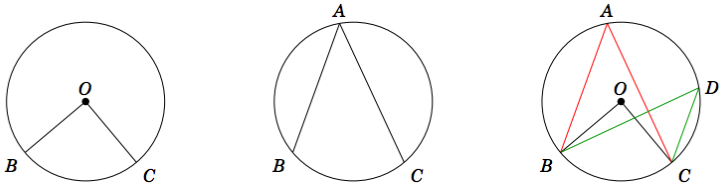
2.5: Circumscribed and Inscribed Circles

Recall from the Law of Sines that any triangle $\triangle ABC$ has a common ratio of sides to sines of opposite angles, namely

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\,.$$

This common ratio has a geometric meaning: it is the diameter (i.e. twice the radius) of the unique circle in which $\triangle ABC$ can be inscribed, called the **circumscribed circle** of the triangle. Before proving this, we need to review some elementary geometry.



(a) Central angle $\angle O$ (b) Inscribed angle $\angle A$ (c) $\angle A = \angle D = \frac{1}{2}\angle O$

Figure 2.5.1 Types of angles in a circle

An **inscribed angle** of a circle is an angle whose vertex is a point A on the circle and whose sides are line segments (called **chords**) from A to two other points on the circle. In Figure 2.5.1(b), $\angle A$ is an inscribed angle that intercepts the arc $\overset{\frown}{BC}$. We state here without proof a useful relation between inscribed and central angles:

Theorem 2.4

If an inscribed angle $\angle A$ and a central angle $\angle O$ intercept the same arc, then $\angle A = \frac{1}{2}\angle O$. Thus, inscribed angles which intercept the same arc are equal.

Figure 2.5.1(c) shows two inscribed angles, $\angle A$ and $\angle D$, which intercept the same arc $\overset{\frown}{BC}$ as the central angle $\angle O$, and hence $\angle A = \angle D = \frac{1}{2}\angle O$ (so $\angle O = 2\angle A = 2\angle D$).

We will now prove our assertion about the common ratio in the Law of Sines:

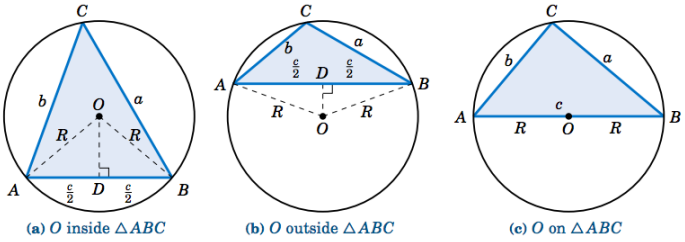
Theorem 2.5

For any triangle $\triangle ABC$, the radius R of its circumscribed circle is given by:

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{2.5.1}$$

Note: For a circle of diameter 1, this means $a = \sin A$, $b = \sin B$, and $c = \sin C$.)

To prove this, let O be the center of the circumscribed circle for a triangle $\triangle ABC$. Then O can be either inside, outside, or on the triangle, as in Figure 2.5.2 below. In the first two cases, draw a perpendicular line segment from O to $\overline{AB}$ at the point D.



(a) O inside $\triangle ABC$ (b) O outside $\triangle ABC$ (c) O on $\triangle ABC$

Figure 2.5.2 Circumscribed circle for $\triangle ABC$

The radii $\overline{OA}$ and $\overline{OB}$ have the same length R, so $\triangle AOB$ is an isosceles triangle. Thus, from elementary geometry we know that $\overline{OD}$ bisects both the angle $\angle AOB$ and the side $\overline{AB}$. So $\angle AOD = \frac{1}{2}\angle AOB$ and $AD = \frac{c}{2}$. But since the inscribed angle $\angle ACB$ and the central angle $\angle AOB$ intercept the same arc $\overset{\frown}{AB}$, we know from Theorem 2.4 that $\angle ACB = \frac{1}{2}\angle AOB$. Hence, $\angle ACB = \angle AOD$. So since $C = \angle ACB$, we have

$$\sin C = \sin \angle AOD = \frac{AD}{OA} = \frac{\frac{c}{2}}{R} = \frac{c}{2R} \quad \Rightarrow \quad 2R = \frac{c}{\sin C},$$

so by the Law of Sines the result follows if O is inside or outside $\triangle ABC$.

Now suppose that O is on $\triangle ABC$, say, on the side $\overline{AB}$, as in Figure 2.5.2(c). Then $\overline{AB}$ is a diameter of the circle, so $C = 90^\circ$ by Thales' Theorem. Hence, $\sin C = 1$, and so $2R = AB = c = \frac{c}{1} = \frac{c}{\sin C}$, and the result again follows by the Law of Sines. **QED**

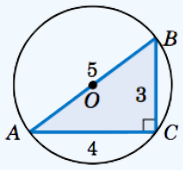


Figure 2.5.3

Solution:

We know that $\triangle ABC$ is a right triangle. So as we see from Figure 2.5.3, $\sin A = 3/5$. Thus,

$$2R = \frac{a}{\sin A} = \frac{3}{\frac{3}{5}} = 5 \quad \Rightarrow \quad \boxed{R = 2.5}.$$

Note that since $R = 2.5$, the diameter of the circle is 5, which is the same as AB. Thus, $\overline{AB}$ must be a diameter of the circle, and so the center O of the circle is the midpoint of $\overline{AB}$.

Corollary 2.6

For any right triangle, the hypotenuse is a diameter of the circumscribed circle, i.e. the center of the circle is the midpoint of the hypotenuse.

For the right triangle in the above example, the circumscribed circle is simple to draw; its center can be found by measuring a distance of 2.5 units from A along $\overline{AB}$.

We need a different procedure for acute and obtuse triangles, since for an acute triangle the center of the circumscribed circle will be inside the triangle, and it will be outside for an obtuse triangle. Notice from the proof of Theorem 2.5 that the center O was on the perpendicular bisector of one of the sides ($\overline{AB}$). Similar arguments for the other sides would show that O is on the perpendicular bisectors for those sides:

Corollary 2.7

For any triangle, the center of its circumscribed circle is the intersection of the perpendicular bisectors of the sides.

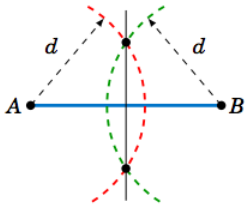


Figure 2.5.4

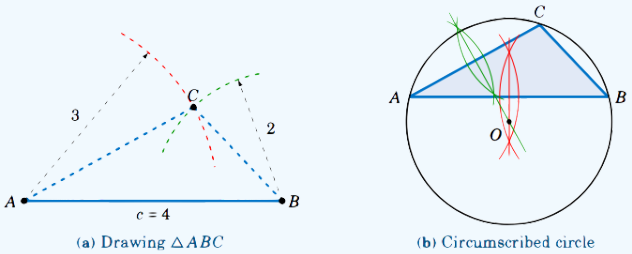
Example 2.18

Find the radius R of the circumscribed circle for the triangle $\triangle ABC$ from Example 2.6 in Section 2.2: $a = 2$, $b = 3$, and $c = 4$. Then draw the triangle and the circle.

Solution:

In Example 2.6 we found $A = 28.9^\circ$, so $2R = \frac{a}{\sin A} = \frac{2}{\sin 28.9^\circ} = 4.14$, so $\boxed{R = 2.07}$.

In Figure 2.5.5(a) we show how to draw $\triangle ABC$: use a ruler to draw the longest side $\overline{AB}$ of length $c = 4$, then use a compass to draw arcs of radius 3 and 2 centered at A and B, respectively. The intersection of the arcs is the vertex C.



(a) Drawing $\triangle ABC$ (b) Circumscribed circle

Figure 2.5.5

In Figure 2.5.5(b) we show how to draw the circumscribed circle: draw the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$; their intersection is the center O of the circle. Use a compass to draw the circle centered at O which passes through A.

Theorem 2.5 can be used to derive another formula for the area of a triangle:

Theorem 2.8

For a triangle $\triangle ABC$, let K be its area and let R be the radius of its circumscribed circle. Then

$$K = \frac{abc}{4R} \quad (\text{and hence } R = \frac{abc}{4K}). \tag{2.5.2}$$

To prove this, note that by Theorem 2.5 we have

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \Rightarrow \quad \sin A = \frac{a}{2R}, \ \sin B = \frac{b}{2R}, \ \sin C = \frac{c}{2R}.$$

Substitute those expressions into Equation 2.26 from Section 2.4 for the area K:

$$K = \frac{a^2 \sin B \sin C}{2 \sin A} = \frac{a^2 \cdot \frac{b}{2R} \cdot \frac{c}{2R}}{2 \cdot \frac{a}{2R}} = \frac{abc}{4R} \qquad \textbf{QED}$$

Combining Theorem 2.8 with Heron's formula for the area of a triangle, we get:

Corollary 2.9

For a triangle $\triangle ABC$, let $s = \frac{1}{2}(a+b+c)$. Then the radius R of its circumscribed circle is

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \,. \tag{2.5.3}$$

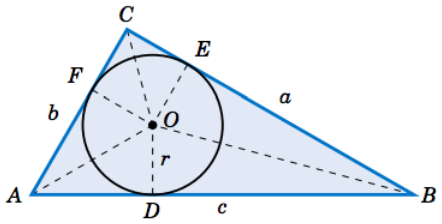


Figure 2.5.6 Inscribed circle for $\triangle ABC$

Let r be the radius of the inscribed circle, and let D, E, and F be the points on $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively, at which the circle is tangent. Then $\overline{OD} \perp \overline{AB}$, $\overline{OE} \perp \overline{BC}$, and $\overline{OF} \perp \overline{AC}$. Thus, $\triangle OAD$ and $\triangle OAF$ are equivalent triangles, since they are right triangles with the same hypotenuse $\overline{OA}$ and with corresponding legs $\overline{OD}$ and $\overline{OF}$ of the same length r. Hence, $\angle OAD = \angle OAF$, which means that $\overline{OA}$ bisects the angle A. Similarly, $\overline{OB}$ bisects B and $\overline{OC}$ bisects C. We have thus shown:

For any triangle, the center of its inscribed circle is the intersection of the bisectors of the angles.

We will use Figure 2.5.6 to find the radius r of the inscribed circle. Since $\overline{OA}$ bisects A, we see that $\tan \frac{1}{2}A = \frac{r}{AD}$, and so $r = AD \cdot \tan \frac{1}{2}A$. Now, $\triangle OAD$ and $\triangle OAF$ are equivalent triangles, so $AD = AF$. Similarly, $DB = EB$ and $FC = CE$. Thus, if we let $s = \frac{1}{2}(a+b+c)$, we see that

$$\begin{aligned} 2s &= a + b + c = (AD+DB) + (CE+EB) + (AF+FC) \\ &= AD + EB + CE + EB + AD + CE = 2(AD+EB+CE) \\ s &= AD + EB + CE = AD + a \\ AD &= s - a \,. \end{aligned}$$

Hence, $r = (s-a)\tan \frac{1}{2}A$. Similar arguments for the angles B and C give us:

Theorem 2.10

For any triangle $\triangle ABC$, let $s = \frac{1}{2}(a+b+c)$. Then the radius r of its inscribed circle is

$$r = (s-a)\tan \tfrac{1}{2}A = (s-b)\tan \tfrac{1}{2}B = (s-c)\tan \tfrac{1}{2}C \,. \tag{2.5.4}$$

We also see from Figure 2.5.6 that the area of the triangle $\triangle AOB$ is

$$\text{Area}(\triangle AOB) = \tfrac{1}{2}\,\text{base} \times \text{height} = \tfrac{1}{2}\,c\,r \,.$$

Similarly, $\text{Area}(\triangle BOC) = \frac{1}{2}\,a\,r$ and $\text{Area}(\triangle AOC) = \frac{1}{2}\,b\,r$. Thus, the area K of $\triangle ABC$ is

$$\begin{aligned} K &= \text{Area}(\triangle AOB) + \text{Area}(\triangle BOC) + \text{Area}(\triangle AOC) = \tfrac{1}{2}\,c\,r + \tfrac{1}{2}\,a\,r + \tfrac{1}{2}\,b\,r \\ &= \tfrac{1}{2}\,(a+b+c)\,r = sr \,, \text{ so by Heron's formula we get} \\ r &= \frac{K}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2}} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \,. \end{aligned}$$

We have thus proved the following theorem:

Theorem 2.11

For any triangle $\triangle ABC$, let $s = \frac{1}{2}(a+b+c)$. Then the radius r of its inscribed circle is

$$r = \frac{K}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \quad . \tag{2.5.5}$$

Recall from geometry how to bisect an angle: use a compass centered at the vertex to draw an arc that intersects the sides of the angle at two points. At those two points use a compass to draw an arc with the same radius, large enough so that the two arcs intersect at a point, as in Figure 2.5.7. The line through that point and the vertex is the bisector of the angle. For the inscribed circle of a triangle, you need only *two* angle bisectors; their intersection will be the center of the circle.

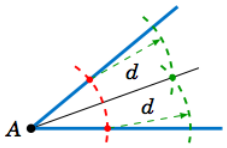


Figure 2.5.7

Example 2.19

Find the radius r of the inscribed circle for the triangle $\triangle ABC$ from Example 2.6 in Section 2.2: $a = 2$, $b = 3$, and $c = 4$. Draw the circle.

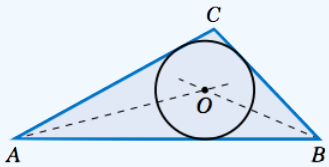


Figure 2.5.8

Solution:

Using Theorem 2.11 with $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(2+3+4) = \frac{9}{2}$, we have

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\frac{\left(\frac{9}{2}-2\right)\left(\frac{9}{2}-3\right)\left(\frac{9}{2}-4\right)}{\frac{9}{2}}} = \sqrt{\frac{5}{12}}\,.$$

Figure 2.5.8 shows how to draw the inscribed circle: draw the bisectors of A and B, then at their intersection use a compass to draw a circle of radius $r = \sqrt{5/12} \approx 0.645$.

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2.E: General Triangles (Exercises)

These are homework exercises to accompany Corral's "Elementary Trigonometry" Textmap. This is a text on elementary trigonometry, designed for students who have completed courses in high-school algebra and geometry. Though designed for college students, it could also be used in high schools. The traditional topics are covered, but a more geometrical approach is taken than usual. Also, some numerical methods (e.g. the secant method for solving trigonometric equations) are discussed.

2.1 Exercises

For Exercises 1-9, solve the triangle $\triangle ABC$.

2.1.1 $a = 10, A = 35^\circ, B = 25^\circ$

2.1.2 $b = 40, B = 75^\circ, c = 35$

2.1.3 $A = 40^\circ, B = 45^\circ, c = 15$

2.1.4 $a = 5, A = 42^\circ, b = 7$

2.1.5 $a = 40, A = 25^\circ, c = 30$

2.1.6 $a = 5, A = 47^\circ, b = 9$

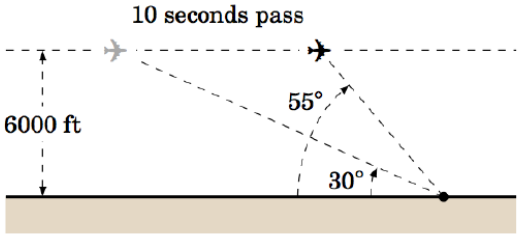
2.1.7 $a = 12, A = 94^\circ, b = 15$

2.1.8 $a = 15, A = 94^\circ, b = 12$

2.1.9 $a = 22, A = 50^\circ, c = 27$

2.1.10 Draw a circle with a radius of 2 inches and inscribe a triangle inside the circle. Use a ruler and a protractor to measure the sides a, b, c and the angles A, B, C of the triangle. The Law of Sines says that the ratios $\frac{a}{\sin A}, \frac{b}{\sin B}, \frac{c}{\sin C}$ are equal. Verify this for your triangle. What relation does that common ratio have to the diameter of your circle?

2.1.11 An observer on the ground measures an angle of inclination of 30° to an approaching airplane, and 10 seconds later measures an angle of inclination of 55°. If the airplane is flying at a constant speed and at a steady altitude of 6000 ft in a straight line directly over the observer, find the speed of the airplane in miles per hour. (Note: 1 mile = 5280 ft)



2.1.12 Prove the Law of Sines for right triangles. (*Hint: One of the angles is known.*)

2.1.13 For a triangle $\triangle ABC$, show that $\frac{a \pm b}{c} = \frac{\sin A \pm \sin B}{\sin C}$.

2.1.14 For a triangle $\triangle ABC$, show that $\frac{a}{c} = \frac{\sin(B+C)}{\sin C}$.

2.1.15 One diagonal of a parallelogram is 17 cm long and makes angles of 36° and 15° with the sides. Find the lengths of the sides.

2.1.16 Explain why in Case 1 (one side and two angles) there is always exactly one solution.

2.2 Exercises

For Exercises 1-6, solve the triangle $\triangle ABC$.

2.2.1 $A = 60^\circ, b = 8, c = 12$

2.2.2 $A = 30^\circ$, $b = 4$, $c = 6$

2.2.3 $a = 7$, $B = 60^\circ$, $c = 9$

2.2.4 $a = 7$, $b = 3$, $c = 9$

2.2.5 $a = 6$, $b = 4$, $c = 1$

2.2.6 $a = 11$, $b = 13$, $c = 16$

2.2.7 The diagonals of a parallelogram intersect at a 42° angle and have lengths of 12 and 7 cm. Find the lengths of the sides of the parallelogram. (*Hint: The diagonals bisect each other*.)

2.2.8 Two trains leave the same train station at the same time, moving along straight tracks that form a 35° angle. If one train travels at an average speed of 100 mi/hr and the other at an average speed of 90 mi/hr, how far apart are the trains after half an hour?

2.2.9 Three circles with radii of 4, 5, and 6 cm, respectively, are tangent to each other externally. Find the angles of the triangle whose vertexes are the centers of the circles.

2.2.10 Find the length x of the diagonal of the quadrilateral in Figure 2.2.3 below.

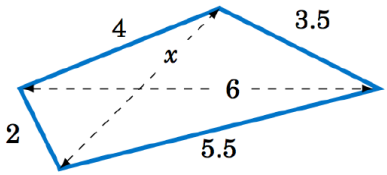


Figure 2.2.3 Exercise 10

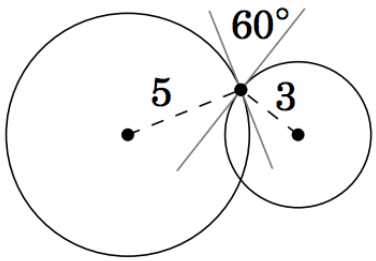


Figure 2.2.4 Exercise 11

2.2.11 Two circles of radii 5 and 3 cm, respectively, intersect at two points. At either point of intersection, the tangent lines to the circles form a 60° angle, as in Figure 2.2.4 above. Find the distance between the centers of the circles.

2.2.12 Use the Law of Cosines to show that for any triangle $\triangle ABC$, $c^2 < a^2 + b^2$ if C is acute, $c^2 > a^2 + b^2$ if C is obtuse, and $c^2 = a^2 + b^2$ if C is a right angle.

2.2.13 Show that for any triangle $\triangle ABC$,

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}.$$

2.2.14 Show that for any triangle $\triangle ABC$,

$$\cos A + \cos B + \cos C = \frac{a^2(b+c-a) + b^2(a+c-b) + c^2(a+b-c)}{2abc}.$$

What do the terms in parentheses represent geometrically? Use your answer to explain why $\cos A + \cos B + \cos C > 0$ for any triangle, even if one of the cosines is negative.

2.2.15 Prove the Law of Cosines (i.e. Equations 2.9-2.11) for right triangles.

2.2.16 Recall from elementary geometry that a *median* of a triangle is a line segment from any vertex to the midpoint of the opposite side. Show that the sum of the squares of the three medians of a triangle is $\frac{3}{4}$ the sum of the squares of the sides.

2.2.17 The Dutch astronomer and mathematician Willebrord Snell (1580-1626) wrote the Law of Cosines as

$$\frac{2ab}{c^2 - (a-b)^2} = \frac{1}{1 - \cos C} \tag{2.E.1}$$

in his trigonometry text *Doctrina triangulorum* (published a year after his death). Show that this formula is equivalent to Equation 2.11 in our statement of the Law of Cosines.

2.2.18 Suppose that a satellite in space, an earth station, and the center of the earth all lie in the same plane. Let r_e be the radius of the earth, let r_s be the distance from the center of the earth to the satellite (called the *orbital radius* of the satellite), and let d be the distance from the earth station to the satellite. Let E be the angle of elevation from the earth station to the satellite, and let γ and ψ be the angles shown in Figure 2.25.

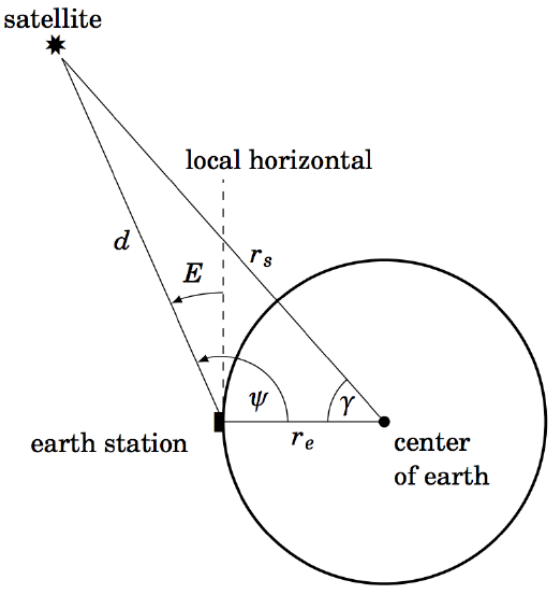


Figure 2.2.5

Use the Law of Cosines to show that

$$d = r_s\sqrt{1 + \left(\frac{r_e}{r_s}\right)^2 - 2\left(\frac{r_e}{r_s}\right)\cos\gamma}\ , \tag{2.E.2}$$

and then use $E = \psi - 90^\circ$ and the Law of Sines to show that

$$\cos E = \frac{\sin \gamma}{\sqrt{1 + \left(\frac{r_e}{r_s}\right)^2 - 2\left(\frac{r_e}{r_s}\right)\cos \gamma}}. \tag{2.E.3}$$

Note: This formula allows the angle of elevation E to be calculated from the coordinates of the earth station and the *subsatellite point* (where the line from the satellite to the center of the earth crosses the surface of the earth).

2.3 Exercises

For Exercises 1-3, use the Law of Tangents to solve the triangle $\triangle ABC$.

2.3.1 $a = 12, b = 8, C = 60^\circ$

2.3.2 $A = 30^\circ, b = 4, c = 6$

2.3.3 $a = 7, B = 60^\circ, c = 9$

For Exercises 4-6, check if it is possible for a triangle to have the given parts.

2.3.4 $a = 5, b = 7, c = 10, A = 27.7^\circ, B = 40.5^\circ, C = 111.8^\circ$

2.3.5 $a = 3, b = 7, c = 9, A = 19.2^\circ, B = 68.2^\circ, C = 92.6^\circ$

2.3.6 $a = 6, b = 9, c = 9, A = 39^\circ, B = 70.5^\circ, C = 70.5^\circ$

2.3.7 Let $\triangle ABC$ be a right triangle with $C = 90^\circ$. Show that $\tan \frac{1}{2}(A - B) = \frac{a-b}{a+b}$.

2.3.8 For any triangle $\triangle ABC$, show that $\tan \frac{1}{2}(A - B) = \frac{a-b}{a+b} \cot \frac{1}{2}C$.

2.3.9 For any triangle $\triangle ABC$, show that $\tan A = \dfrac{a \sin B}{c - a \cos B}$. (*Hint: Draw the altitude from the vertex C to $\overline{AB}$.*) Notice that this formula provides another way of solving a triangle in Case 3 (two sides and the included angle).

2.3.10 For any triangle $\triangle ABC$, show that $c = b \cos A + a \cos B$. This is another check of a triangle.

2.3.11 If $b \cos A = a \cos B$, show that the triangle $\triangle ABC$ is isosceles.

2.3.12 Let $ABCD$ be a quadrilateral which completely contains its two diagonals. The quadrilateral has eight parts: four sides and four angles. What is the smallest number of parts that you would need to know to solve the quadrilateral? Explain your answer.

2.4 Exercises

For Exercises 1-6, find the area of the triangle $\triangle ABC$.

2.4.1 $A = 70^\circ, b = 4, c = 12$

2.4.2 $a = 10, B = 95^\circ, c = 35$

2.4.3 $A = 10^\circ, B = 48^\circ, C = 122^\circ, c = 11$

2.4.4 $A = 171^\circ, B = 1^\circ, C = 8^\circ, b = 2$

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2.4.5 $a = 2$, $b = 3$, $c = 4$

2.4.6 $a = 5$, $b = 6$, $c = 5$

2.4.7 Find the area of the quadrilateral in Figure 2.4.3 below.

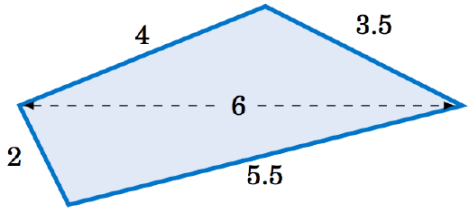

Figure 2.4.3 Exercise 7

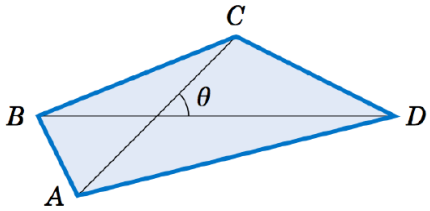

Figure 2.4.4 Exercise 8

2.4.8 Let $ABCD$ be a quadrilateral which completely contains its two diagonals, as in Figure 2.4.4 above. Show that the area K of $ABCD$ is equal to half the product of its diagonals and the sine of the angle they form, i.e. $K = \frac{1}{2}\, AC \cdot BD \, \sin\, \theta$.

2.4.9 From Equation 2.26 derive the following formula for the area of a triangle $\triangle ABC$:

$$\text{Area} \;=\; K \;=\; \frac{a^2 \,\sin\, B \,\sin\, C}{2 \,\sin\,(B+C)} \tag{2.E.4}$$

2.4.10 Show that the triangle area formula

$$\text{Area} \;=\; K \;=\; \tfrac{1}{4}\sqrt{\,(a+(b+c))\,(c-(a-b))\,(c+(a-b))\,(a+(b-c))\,} \tag{2.E.5}$$

is equivalent to Heron's formula. (*Hint: In Heron's formula replace* s by $\frac{1}{2}(a+b+c)$.)

2.4.11 Show that the triangle area Equation 2.34 is equivalent to Heron's formula. (*Hint: Factor the expression inside the square root.*)

2.4.12 Find the angle A in Example 2.16, then use Equation 2.23 to find the area. Did it work?

2.5 Exercises

For Exercises 1-6, find the radii R and r of the circumscribed and inscribed circles, respectively, of the triangle $\triangle ABC$.

2.5.1 $a = 2$, $b = 4$, $c = 5$

2.5.2 $a = 6$, $b = 8$, $c = 8$

2.5.3 $a = 5$, $b = 7$, $C = 40^\circ$

2.5.4 $A = 170^\circ$, $b = 100$, $c = 300$

2.5.5 $a = 10, b = 11, c = 20.5$

2.5.6 $a = 5, b = 12, c = 13$

For Exercises 7 and 8, draw the triangle $\triangle ABC$ and its circumscribed and inscribed circles accurately, using a ruler and compass (or computer software).

2.5.7 $a = 2$ in, $b = 4$ in, $c = 5$ in

2.5.8 $a = 5$ in, $b = 6$ in, $c = 7$ in

2.5.9 For any triangle $\triangle ABC$, let $s = \frac{1}{2}(a+b+c)$. Show that

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\ , \quad \tan\tfrac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\ , \quad \tan\tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\ . \qquad (2.E.6)$$

2.5.10 Show that for any triangle $\triangle ABC$, the radius R of its circumscribed circle is

$$R = \frac{abc}{\sqrt{(a+b+c)(b+c-a)(a-b+c)(a+b-c)}}\ . \qquad (2.E.7)$$

2.5.11 Show that for any triangle $\triangle ABC$, the radius R of its circumscribed circle and the radius r of its inscribed circle satisfy the relation

$$rR = \frac{abc}{2(a+b+c)}\ . \qquad (2.E.8)$$

2.5.12 Let $\triangle ABC$ be an equilateral triangle whose sides are of length a.

(a) Find the exact value of the radius R of the circumscribed circle of $\triangle ABC$.
(b) Find the exact value of the radius r of the inscribed circle of $\triangle ABC$.
(c) How much larger is R than r?
(d) Show that the circumscribed and inscribed circles of $\triangle ABC$ have the same center.

2.5.13 Let $\triangle ABC$ be a right triangle with $C = 90^\circ$. Show that $\tan\frac{1}{2}A = \sqrt{\frac{c-b}{c+b}}$.

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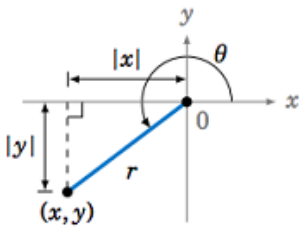
CHAPTER OVERVIEW

3: IDENTITIES

3.1: BASIC TRIGONOMETRIC IDENTITIES

Equations that are true for angles θ for which both sides of the equation are defined are called identities. In this section we will discuss several identities involving the trigonometric functions that are often used to simplify complicated expressions or equations.



3.2: SUM AND DIFFERENCE FORMULAS

We will now derive identities for the trigonometric functions of the sum and difference of two angles.

3.3: DOUBLE-ANGLE AND HALF-ANGLE FORMULAS

A special case of the addition formulas is when the two angles being added are equal, resulting in the double-angle formulas.

3.4: OTHER IDENTITIES

Though the identities in this section fall under the category of “other”, they are perhaps (along with $\cos^2\theta + \sin^2\theta = 1$) the most widely used identities in practice. It is very common to encounter terms such as $\sin A + sinB$ or $\sin A \cos B$ in calculations, so we will now derive identities for those expressions.

3.E: IDENTITIES (EXERCISES)

These are homework exercises to accompany Corral's "Elementary Trigonometry" Textmap. This is a text on elementary trigonometry, designed for students who have completed courses in high-school algebra and geometry. Though designed for college students, it could also be used in high schools. The traditional topics are covered, but a more geometrical approach is taken than usual. Also, some numerical methods (e.g. the secant method for solving trigonometric equations) are discussed.

3.1: Basic Trigonometric Identities

So far we know a few relations between the trigonometric functions. For example, we know the reciprocal relations:

1. $\csc\ \theta = \dfrac{1}{\sin\ \theta}$ when $\sin\ \theta \neq 0$
2. $\sec\ \theta = \dfrac{1}{\cos\ \theta}$ when $\cos\ \theta \neq 0$
3. $\cot\ \theta = \dfrac{1}{\tan\ \theta}$ when $\tan\ \theta$ is defined and not 0
4. $\sin\ \theta = \dfrac{1}{\csc\ \theta}$ when $\csc\ \theta$ is defined and not 0
5. $\cos\ \theta = \dfrac{1}{\sec\ \theta}$ when $\sec\ \theta$ is defined and not 0
6. $\tan\ \theta = \dfrac{1}{\cot\ \theta}$ when $\cot\ \theta$ is defined and not 0

Notice that each of these equations is true for *all* angles θ for which both sides of the equation are defined. Such equations are called **identities**, and in this section we will discuss several *trigonometric identities*, i.e. identities involving the trigonometric functions. These identities are often used to simplify complicated expressions or equations. For example, one of the most useful trigonometric identities is the following:

$$\tan\ \theta = \frac{\sin\ \theta}{\cos\ \theta} \qquad \text{when } \cos\ \theta \neq 0 \tag{3.1.1}$$

To prove this identity, pick a point (x, y) on the terminal side of θ a distance $r > 0$ from the origin, and suppose that $\cos\ \theta \neq 0$. Then $x \neq 0$ (since $\cos\ \theta = \frac{x}{r}$), so by definition

$$\frac{\sin\ \theta}{\cos\ \theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan\ \theta\,.$$

Note how we proved the identity by expanding one of its sides ($\frac{\sin\ \theta}{\cos\ \theta}$) until we got an expression that was equal to the other side ($\tan\ \theta$). This is probably the most common technique for proving identities. Taking reciprocals in the above identity gives:

$$\cot\ \theta = \frac{\cos\ \theta}{\sin\ \theta} \qquad \text{when } \sin\ \theta \neq 0 \tag{3.1.2}$$

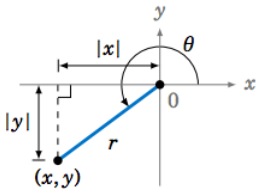


Figure 3.1.1

We will now derive one of the most important trigonometric identities. Let θ be any angle with a point (x, y) on its terminal side a distance $r > 0$ from the origin. By the Pythagorean Theorem, $r^2 = x^2 + y^2$ (and hence $r = \sqrt{x^2 + y^2}$). For example, if θ is in QIII as in Figure 3.1.1, then the legs of the right triangle formed by the reference angle have lengths $|x|$ and $|y|$ (we use absolute values because x and y are negative in QIII). The same argument holds if θ is in the other quadrants or on either axis. Thus,

$$r^2 = |x|^2 + |y|^2 = x^2 + y^2\,,$$

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so dividing both sides of the equation by r^2 (which we can do since $r > 0$) gives

$$\frac{r^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2.$$

Since $\frac{r^2}{r^2} = 1$, $\frac{x}{r} = \cos\ \theta$, and $\frac{y}{r} = \sin\ \theta$, we can rewrite this as:

$$\cos^2\ \theta + \sin^2\ \theta = 1 \tag{3.1.3}$$

You can think of this as sort of a trigonometric variant of the Pythagorean Theorem. Note that we use the notation $\sin^2\ \theta$ to mean $(\sin\ \theta)^2$, likewise for cosine and the other trigonometric functions. We will use the same notation for other powers besides 2.

From the above identity we can derive more identities. For example:

$$\sin^2\ \theta = 1 - \cos^2\ \theta \tag{3.1.4}$$

$$\cos^2\ \theta = 1 - \sin^2\ \theta \tag{3.1.5}$$

from which we get (after taking square roots):

$$\sin\ \theta = \pm\sqrt{1 - \cos^2\ \theta} \tag{3.1.6}$$

$$\cos\ \theta = \pm\sqrt{1 - \sin^2\ \theta} \tag{3.1.7}$$

Also, from the inequalities $0 \leq \sin^2\ \theta = 1 - \cos^2\ \theta \leq 1$ and $0 \leq \cos^2\ \theta = 1 - \sin^2\ \theta \leq 1$, taking square roots gives us the following bounds on sine and cosine:

$$-1 \leq \sin\ \theta \leq 1 \tag{3.1.8}$$

$$-1 \leq \cos\ \theta \leq 1 \tag{3.1.9}$$

The above inequalities are not identities (since they are not equations), but they provide useful checks on calculations. Recall that we derived those inequalities from the definitions of sine and cosine in Section 1.4.

In Equation 3.1.3, dividing both sides of the identity by $\cos^2\ \theta$ gives

$$\frac{\cos^2\ \theta}{\cos^2\ \theta} + \frac{\sin^2\ \theta}{\cos^2\ \theta} = \frac{1}{\cos^2\ \theta},$$

so since $\tan\ \theta = \frac{\sin\ \theta}{\cos\ \theta}$ and $\sec\ \theta = \frac{1}{\cos\ \theta}$, we get:

$$1 + \tan^2\ \theta = \sec^2\ \theta \tag{3.1.10}$$

Likewise, dividing both sides of Equation 3.1.3 by $\sin^2\ \theta$ gives

$$\frac{\cos^2\ \theta}{\sin^2\ \theta} + \frac{\sin^2\ \theta}{\sin^2\ \theta} = \frac{1}{\sin^2\ \theta},$$

so since $\cot\ \theta = \frac{\cos\ \theta}{\sin\ \theta}$ and $\csc\ \theta = \frac{1}{\sin\ \theta}$, we get:

$$\cot^2\ \theta + 1 = \csc^2\ \theta \tag{3.1.11}$$

Example 3.1

Simplify $\cos^2\theta\tan^2\theta$.

Solution

We can use Equation 3.1.5 to simplify:

$$\begin{aligned}\cos^2\theta\tan^2\theta &= \cos^2\theta\cdot\frac{\sin^2\theta}{\cos^2\theta}\\ &= \sin^2\theta\end{aligned}$$

Example 3.2

Simplify $5\sin^2\theta+4\cos^2\theta$.

Solution

We can use Equation 3.1.1 to simplify:

$$\begin{aligned}5\sin^2\theta+4\cos^2\theta &= 5\sin^2\theta+4\left(1-\sin^2\theta\right)\\ &= 5\sin^2\theta+4-4\sin^2\theta\\ &= \sin^2\theta+4\end{aligned}$$

Example 3.3

Prove that $\tan\theta+\cot\theta=\sec\theta\csc\theta$.

Solution

We will expand the left side and show that it equals the right side:

$$\begin{aligned}\tan\theta+\cot\theta &= \frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta} && \text{(by 3.1.1 and 3.1.2)}\\ &= \frac{\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\sin\theta}+\frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{\cos\theta} && \text{(multiply both fractions by 1)}\\ &= \frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta} && \text{(after getting a common denominator)}\\ &= \frac{1}{\cos\theta\sin\theta} && \text{(by 3.1.3)}\\ &= \frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}\\ &= \sec\theta\csc\theta\end{aligned}$$

In the above example, how did we know to expand the left side instead of the right side? In general, though this technique does not always work, the more complicated side of the identity is likely to be easier to expand. The reason is that, by its complexity, there will be more things that you can do with that expression. For example, if you were asked to prove that

$$\sec\theta-\sin\theta\tan\theta=\cos\theta,$$

there would not be much that you could do with the right side of that identity; it consists of a single term ($\cos\theta$) that offers no obvious means of expansion.

Example 3.4

Prove that $\dfrac{1+\cot^2\theta}{\sec\theta}=\csc\theta\cot\theta$.

Solution

Of the two sides, the left side looks more complicated, so we will expand that:

$$\begin{aligned}\frac{1+\cot^2\theta}{\sec\theta} &= \frac{\csc^2\theta}{\sec\theta} && \text{(by 3.1.11)}\\ &= \frac{\csc\theta\cdot\dfrac{1}{\sin\theta}}{\dfrac{1}{\cos\theta}}\\ &= \csc\theta\cdot\frac{\cos\theta}{\sin\theta}\\ &= \csc\theta\cot\theta && \text{(by 3.1.2)}\end{aligned}$$

When trying to prove an identity where at least one side is a ratio of expressions, *cross-multiplying* can be an effective technique:

$$\frac{a}{b}=\frac{c}{d} \quad \text{if and only if} \quad ad=bc$$

Example 3.6

Prove that $\dfrac{1+\sin\theta}{\cos\theta}=\dfrac{\cos\theta}{1-\sin\theta}$.

Solution

Cross-multiply and reduce both sides until it is clear that they are equal:

$$\begin{aligned}(1+\sin\theta)(1-\sin\theta) &= \cos\theta\cdot\cos\theta\\ 1-\sin^2\theta &= \cos^2\theta\end{aligned}$$

By 3.1.5 both sides of the last equation are indeed equal. Thus, the original identity holds.

Example 3.7

Suppose that $a\cos\theta=b$ and $c\sin\theta=d$ for some angle θ and some constants a, b, c, and d. Show that $a^2c^2=b^2c^2+a^2d^2$.

Solution

Multiply both sides of the first equation by c and the second equation by a:

$$\begin{aligned}ac\cos\theta &= bc\\ ac\sin\theta &= ad\end{aligned}$$

Now square each of the above equations then add them together to get:

$$\begin{aligned}(ac\cos\theta)^2+(ac\sin\theta)^2 &= (bc)^2+(ad)^2\\ (ac)^2\left(\cos^2\theta+\sin^2\theta\right) &= b^2c^2+a^2d^2\\ a^2c^2 &= b^2c^2+a^2d^2 && \text{(by 3.1.3)}\end{aligned}$$

Notice how θ does not appear in our final result. The trick was to get a common coefficient (ac) for $\cos\theta$ and $\sin\theta$ so that we could use $\cos^2\theta+\sin^2\theta=1$. This is a common technique for eliminating trigonometric functions from systems of equations.

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3.2: Sum and Difference Formulas

We will now derive identities for the trigonometric functions of the sum and difference of two angles. For the sum of any two angles A and B, we have the *addition formulas:*

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \tag{3.2.1}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \tag{3.2.2}$$

To prove these, first assume that A and B are acute angles. Then $A+B$ is either acute or obtuse, as in Figure 3.2.1. Note in both cases that $\angle QPR = A$, since

$$\angle QPR = \angle QPO - \angle OPM = (90^\circ - B) - (90^\circ - (A+B)) = A \text{ in Figure 3.2.1(a), and}$$

$$\angle QPR = \angle QPO + \angle OPM = (90^\circ - B) + (90^\circ - (180^\circ - (A+B))) = A \text{ in Figure 3.2.1(b).}$$

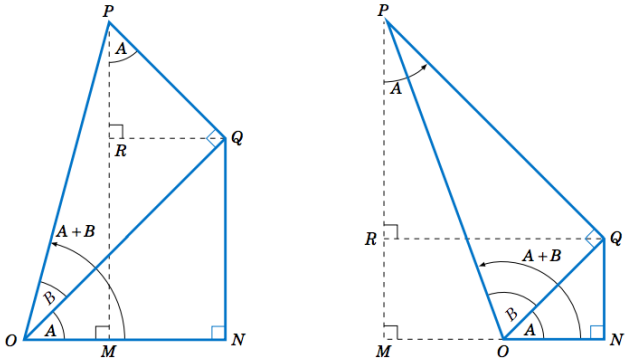

Figure 3.2.1 $\sin(A+B)$ and $\cos(A+B)$ for acute A and B

Thus,

$$\begin{aligned}\sin(A+B) &= \frac{MP}{OP} = \frac{MR+RP}{OP} = \frac{NQ+RP}{OP} = \frac{NQ}{OP} + \frac{RP}{OP} \\ &= \frac{NQ}{OQ}\cdot\frac{OQ}{OP} + \frac{RP}{PQ}\cdot\frac{PQ}{OP} \\ &= \sin A \cos B + \cos A \sin B\,, \end{aligned} \tag{3.2.3}$$

and

$$\begin{aligned}\cos(A+B) &= \frac{OM}{OP} = \frac{ON-MN}{OP} = \frac{ON-RQ}{OP} = \frac{ON}{OP} - \frac{RQ}{OP} \\ &= \frac{ON}{OQ}\cdot\frac{OQ}{OP} - \frac{RQ}{PQ}\cdot\frac{PQ}{OP} \\ &= \cos A \cos B - \sin A \sin B\,. \end{aligned} \tag{3.2.4}$$

So we have proved the identities for acute angles A and B. It is simple to verify that they hold in the special case of $A = B = 0^\circ$. For general angles, we will need to use the relations we derived in Section 1.5 which involve adding or subtracting 90°:

$$\sin(\theta+90^\circ) = \cos\theta \qquad \sin(\theta-90^\circ) = -\cos\theta$$

$$\cos(\theta+90^\circ) = -\sin\theta \qquad \cos(\theta-90^\circ) = \sin\theta$$

These will be useful because *any angle can be written as the sum of an acute angle* (*or* 0°) *and integer multiples of* $\pm 90^\circ$. For example, $155^\circ = 65^\circ + 90^\circ$, $222^\circ = 42^\circ + 2(90^\circ)$, $-77^\circ = 13^\circ - 90^\circ$, etc. So if we can prove that the identities hold when adding or subtracting 90° to or from either A or B, respectively, where A and B are acute or 0°, then the identities will also

hold when repeatedly adding or subtracting 90°, and hence will hold for all angles. Replacing A by $A+90^\circ$ and using the relations for adding 90° gives

$$\begin{aligned}\sin\left((A+90^\circ)+B\right) &= \sin\left((A+B)+90^\circ\right) = \cos\left(A+B\right),\\ &= \cos A\cos B - \sin A\sin B \text{ (by Equation 3.2.4)}\\ &= \sin\left(A+90^\circ\right)\cos B + \cos\left(A+90^\circ\right)\sin B,\end{aligned}$$

so the identity holds for $A+90^\circ$ and B (and, similarly, for A and $B+90^\circ$). Likewise,

$$\begin{aligned}\sin\left((A-90^\circ)+B\right) &= \sin\left((A+B)-90^\circ\right) = -\cos\left(A+B\right),\\ &= -(\cos A\cos B - \sin A\sin B)\\ &= (-\cos A)\cos B + \sin A\sin B\\ &= \sin\left(A-90^\circ\right)\cos B + \cos\left(A-90^\circ\right)\sin B,\end{aligned}$$

so the identity holds for $A-90^\circ$ and B (and, similarly, for A and $B+90^\circ$). Thus, the addition Equation 3.2.1 for sine holds for *all* A and B. A similar argument shows that the addition Equation 3.2.2 for cosine is true for all A and B. **QED**

Replacing B by $-B$ in the addition formulas and using the relations $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$ from Section 1.5 gives us the *subtraction formulas*:

$$\sin\left(A-B\right) = \sin A\cos B - \cos A\sin B \tag{3.2.5}$$

$$\cos\left(A-B\right) = \cos A\cos B + \sin A\sin B \tag{3.2.6}$$

Using the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$, and the addition formulas for sine and cosine, we can derive the addition formula for tangent:

$$\begin{aligned}\tan\left(A+B\right) &= \frac{\sin\left(A+B\right)}{\cos\left(A+B\right)}\\ &= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}\\ &= \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}} \quad \text{(divide top and bottom by } \cos A\cos B)\\ &= \frac{\dfrac{\sin A}{\cos A}\cdot\dfrac{\cos B}{\cancel{\cos B}} + \dfrac{\cos A}{\cancel{\cos A}}\cdot\dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A}\cdot\dfrac{\sin B}{\cos B}} = \frac{\tan A + \tan B}{1 - \tan A\tan B}\end{aligned}$$

This, combined with replacing B by $-B$ and using the relation $\tan(-\theta) = -\tan\theta$, gives us the addition and subtraction formulas for tangent:

$$\tan\left(A+B\right) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \tag{3.2.7}$$

$$\tan\left(A-B\right) = \frac{\tan A - \tan B}{1 + \tan A\tan B} \tag{3.2.8}$$

Example 3.8

Given angles A and B such that $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{5}$, $\sin B = \frac{12}{13}$, and $\cos B = \frac{5}{13}$, find the exact values of $\sin\left(A+B\right)$, $\cos\left(A+B\right)$, and $\tan\left(A+B\right)$.

Solution

Using the addition formula for sine, we get:

$$\begin{aligned}\sin(A+B) &= \sin A\cos B+\cos A\sin B\\ &= \frac{4}{5}\cdot\frac{5}{13}+\frac{3}{5}\cdot\frac{12}{13} \quad\Rightarrow\quad \boxed{\sin(A+B)=\frac{56}{65}}\end{aligned}$$

Using the addition formula for cosine, we get:

$$\begin{aligned}\cos(A+B) &= \cos A\cos B-\sin A\sin B\\ &= \frac{3}{5}\cdot\frac{5}{13}-\frac{4}{5}\cdot\frac{12}{13} \quad\Rightarrow\quad \boxed{\cos(A+B)=-\frac{33}{65}}\end{aligned}$$

Instead of using the addition formula for tangent, we can use the results above:

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\frac{56}{65}}{-\frac{33}{65}} \quad\Rightarrow\quad \boxed{\tan(A+B)=-\frac{56}{33}}$$

Example 3.9

Prove the following identity:

$$\sin(A+B+C)=\sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C-\sin A\sin B\sin C$$

Solution

Treat $A+B+C$ as $(A+B)+C$ and use the addition formulas three times:

$$\begin{aligned}\sin(A+B+C) &= \sin((A+B)+C)\\ &= \sin(A+B)\cos C+\cos(A+B)\sin C\\ &= (\sin A\cos B+\cos A\sin B)\cos C+(\cos A\cos B-\sin A\sin B)\sin C\\ &= \sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C-\sin A\sin B\sin C\end{aligned}$$

Example 3.10

For any triangle $\triangle ABC$, show that $\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

Solution

Note that this is *not* an identity which holds for *all* angles; since A, B, and C are the angles of a triangle, it holds when A, B, $C>0^\circ$ and $A+B+C=180^\circ$. So using $C=180^\circ-(A+B)$ and the relation $\tan(180^\circ-\theta)=-\tan\theta$ from Section 1.5, we get:

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$$\begin{aligned}
\tan A + \tan B + \tan C &= \tan A + \tan B + \tan(180^\circ - (A+B)) \\
&= \tan A + \tan B - \tan(A+B) \\
&= \tan A + \tan B - \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
&= (\tan A + \tan B)\left(1 - \frac{1}{1 - \tan A \tan B}\right) \\
&= (\tan A + \tan B)\left(\frac{1 - \tan A \tan B}{1 - \tan A \tan B} - \frac{1}{1 - \tan A \tan B}\right) \\
&= (\tan A + \tan B) \cdot \left(\frac{-\tan A \tan B}{1 - \tan A \tan B}\right) \\
&= \tan A \tan B \cdot \left(-\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) \\
&= \tan A \tan B \cdot (-\tan(A+B)) \\
&= \tan A \tan B \cdot (\tan(180^\circ - (A+B))) \\
&= \tan A \tan B \tan C
\end{aligned}$$

Example 3.11

Let A, B, C, and D be positive angles such that $A+B+C+D = 180^\circ$. Show that

$$\sin A \sin B + \sin C \sin D = \sin(A+C)\sin(B+C).$$

Solution

It may be tempting to expand the right side, since it appears more complicated. However, notice that the right side has no D term. So instead, we will expand the left side, since we can eliminate the D term on that side by using $D = 180^\circ - (A+B+C)$ and the relation

$$\sin(180^\circ - (A+B+C)) = \sin(A+B+C).$$

So since $\sin D = \sin(A+B+C)$, we get

$$\begin{aligned}
\sin A \sin B + \sin C \sin D &= \sin A \sin B + \sin C \sin(A+B+C)\text{, so by Example 3.9 we get} \\
&= \sin A \sin B + \sin C (\sin A \cos B \cos C + \cos A \sin B \cos C \\
&\quad + \cos A \cos B \sin C - \sin A \sin B \sin C) \\
&= \sin A \sin B + \sin C \sin A \cos B \cos C + \sin C \cos A \sin B \cos C \\
&\quad + \sin C \cos A \cos B \sin C - \sin C \sin A \sin B \sin C .
\end{aligned}$$

It may not be immediately obvious where to go from here, but it is not completely guesswork. We need to end up with $\sin(A+C)\sin(B+C)$, and we know that $\sin(B+C) = \sin B \cos C + \cos B \sin C$. There are two terms involving $\cos B \sin C$, so group them together to get

$$\begin{aligned}
\sin A \sin B + \sin C \sin D &= \sin A \sin B - \sin C \sin A \sin B \sin C + \sin C \cos A \sin B \cos C \\
&\quad + \cos B \sin C (\sin A \cos C + \cos A \sin C) \\
&= \sin A \sin B (1 - \sin^2 C) + \sin C \cos A \sin B \cos C \\
&\quad + \cos B \sin C \sin(A+C) \\
&= \sin A \sin B \cos^2 C + \sin C \cos A \sin B \cos C \\
&\quad + \cos B \sin C \sin(A+C) .
\end{aligned}$$

We now have two terms involving $\sin B \cos C$, which we can factor out:

$$\begin{aligned}\sin A \sin B + \sin C \sin D &= \sin B \cos C (\sin A \cos C + \cos A \sin C) \\ &\quad + \cos B \sin C \sin (A+C) \\ &= \sin B \cos C \sin (A+C) + \cos B \sin C \sin (A+C) \\ &= \sin (A+C)(\sin B \cos C + \cos B \sin C) \\ &= \sin (A+C) \sin (B+C)\end{aligned}$$

Example 3.12

In the study of the propagation of electromagnetic waves, *Snell's law* gives the relation

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 , \tag{3.2.9}$$

where θ_1 is the *angle of incidence* at which a wave strikes the planar boundary between two mediums, θ_2 is the *angle of transmission* of the wave through the new medium, and n_1 and n_2 are the *indexes of refraction* of the two mediums. The quantity

$$r_{1\,2\,s} = \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2} \tag{3.2.10}$$

is called the *Fresnel coefficient* for normal incidence reflection of the wave for *s-polarization*. Show that this can be written as:

$$r_{1\,2\,s} = \frac{\sin (\theta_2 - \theta_1)}{\sin (\theta_2 + \theta_1)}$$

Solution

Multiply the top and bottom of $r_{1\,2\,s}$ by $\sin \theta_1 \sin \theta_2$ to get:

$$\begin{aligned} r_{1\,2\,s} &= \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2} \cdot \frac{\sin \theta_1 \sin \theta_2}{\sin \theta_1 \sin \theta_2} \\ &= \frac{(n_1 \sin \theta_1) \sin \theta_2 \cos \theta_1 - (n_2 \sin \theta_2) \cos \theta_2 \sin \theta_1}{(n_1 \sin \theta_1) \sin \theta_2 \cos \theta_1 + (n_2 \sin \theta_2) \cos \theta_2 \sin \theta_1} \\ &= \frac{(n_1 \sin \theta_1) \sin \theta_2 \cos \theta_1 - (n_1 \sin \theta_1) \cos \theta_2 \sin \theta_1}{(n_1 \sin \theta_1) \sin \theta_2 \cos \theta_1 + (n_1 \sin \theta_1) \cos \theta_2 \sin \theta_1} \qquad \text{(by Snell's law)} \\ &= \frac{\sin \theta_2 \cos \theta_1 - \cos \theta_2 \sin \theta_1}{\sin \theta_2 \cos \theta_1 + \cos \theta_2 \sin \theta_1} \\ &= \frac{\sin (\theta_2 - \theta_1)}{\sin (\theta_2 + \theta_1)} \end{aligned}$$

The last two examples demonstrate an important aspect of how identities are used in practice: recognizing terms which are part of known identities, so that they can be factored out. This is a common technique.

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3.3: Double-Angle and Half-Angle Formulas

A special case of the addition formulas is when the two angles being added are equal, resulting in the *double-angle formulas*:

$$\sin 2\theta = 2 \sin\theta \cos\theta \tag{3.3.1}$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{3.3.2}$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{3.3.3}$$

To derive the sine double-angle formula, we see that

$$\sin 2\theta = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta.$$

Likewise, for the cosine double-angle formula, we have

$$\cos 2\theta = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta,$$

and for the tangent we get

$$\tan 2\theta = \tan(\theta+\theta) = \frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta} = \frac{2\tan\theta}{1-\tan^2\theta}$$

Using the identities $\sin^2\theta = 1-\cos^2\theta$ and $\cos^2\theta = 1-\sin^2\theta$, we get the following useful alternate forms for the cosine double-angle formula:

$$\cos 2\theta = 2\cos^2\theta - 1 \tag{3.3.4}$$
$$= 1 - 2\sin^2\theta \tag{3.3.5}$$

Using $3\theta = 2\theta + \theta$, the addition Equation for sine, and the double-angle Equations 3.3.1 and 3.3.5, we get:

$$\begin{aligned}
\sin 3\theta &= \sin(2\theta+\theta) \\
&= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \\
&= (2\sin\theta\cos\theta)\cos\theta + (1-2\sin^2\theta)\sin\theta \\
&= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta \\
&= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta \\
&= 3\sin\theta - 4\sin^3\theta
\end{aligned}$$

Expand the right side and use $1+\tan^2 z = \sec^2 z$:

$$\begin{aligned}
\frac{4\tan z(1-\tan^2 z)}{(1+\tan^2 z)^2} &= \frac{4\cdot\dfrac{\sin z}{\cos z}\cdot\left(\dfrac{\cos^2 z}{\cos^2 z} - \dfrac{\sin^2 z}{\cos^2 z}\right)}{(\sec^2 z)^2} \\
&= \frac{4\cdot\dfrac{\sin z}{\cos z}\cdot\dfrac{\cos 2z}{\cos^2 z}}{\left(\dfrac{1}{\cos^2 z}\right)^2} \qquad \text{(by Equation 3.3.2)} \\
&= (4\sin z\cos 2z)\cos z \\
&= 2(2\sin z\cos z)\cos 2z \\
&= 2\sin 2z\cos 2z \qquad \text{(by Equation 3.3.1)} \\
&= \sin 4z \qquad \text{(by Equation 3.3.1 with } \theta \text{ replaced by } 2z)
\end{aligned}$$

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Note: Perhaps surprisingly, this seemingly obscure identity has found a use in physics, in the derivation of a solution of the *sine-Gordon equation* in the theory of nonlinear waves

Closely related to the double-angle formulas are the *half-angle formulas*:

$$\sin^2 \tfrac{1}{2}\theta = \frac{1 - \cos\theta}{2} \tag{3.3.6}$$

$$\cos^2 \tfrac{1}{2}\theta = \frac{1 + \cos\theta}{2} \tag{3.3.7}$$

$$\tan^2 \tfrac{1}{2}\theta = \frac{1 - \cos\theta}{1 + \cos\theta} \tag{3.3.8}$$

These formulas are just the double-angle formulas rewritten with θ replaced by $\frac{1}{2}\theta$:

$$\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2} \Rightarrow \sin^2\tfrac{1}{2}\theta = \frac{1-\cos 2(\frac{1}{2}\theta)}{2} = \frac{1-\cos\theta}{2}$$

$$\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2} \Rightarrow \cos^2\tfrac{1}{2}\theta = \frac{1+\cos 2(\frac{1}{2}\theta)}{2} = \frac{1+\cos\theta}{2}$$

The tangent half-angle Equation then follows easily:

$$\tan^2\tfrac{1}{2}\theta = \left(\frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\right)^2 = \frac{\sin^2\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta} = \frac{\frac{1-\cos\theta}{2}}{\frac{1+\cos\theta}{2}} = \frac{1-\cos\theta}{1+\cos\theta}$$

The half-angle formulas are often used (e.g. in calculus) to replace a squared trigonometric function by a nonsquared function, especially when 2θ is used instead of θ.

By taking square roots, we can write the above formulas in an alternate form:

$$\sin\tfrac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{2}} \tag{3.3.9}$$

$$\cos\tfrac{1}{2}\theta = \pm\sqrt{\frac{1+\cos\theta}{2}} \tag{3.3.10}$$

$$\tan\tfrac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \tag{3.3.11}$$

In the above form, the sign in front of the square root is determined by the quadrant in which the angle $\frac{1}{2}\theta$ is located. For example, if $\theta = 300^\circ$ then $\frac{1}{2}\theta = 150^\circ$ is in QII. So in this case $\cos\frac{1}{2}\theta < 0$ and hence we would have $\cos\frac{1}{2}\theta = -\sqrt{\frac{1+\cos\theta}{2}}$.

In Equation 3.3.11, multiplying the numerator and denominator inside the square root by $(1-\cos\theta)$ gives

$$\tan\tfrac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta}} = \pm\sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}} = \pm\sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} = \pm\frac{1-\cos\theta}{\sin\theta}.$$

But $1-\cos\theta \geq 0$, and it turns out (see Exercise 10) that $\tan\frac{1}{2}\theta$ and $\sin\theta$ always have the same sign. Thus, the minus sign in front of the last expression is not possible (since that would switch the signs of $\tan\frac{1}{2}\theta$ and $\sin\theta$), so we have:

$$\tan\tfrac{1}{2}\theta = \frac{1-\cos\theta}{\sin\theta} \tag{3.3.12}$$

Multiplying the numerator and denominator in Equation 3.3.12 by $1+\cos\theta$ gives

$$\tan\tfrac{1}{2}\theta = \frac{1-\cos\theta}{\sin\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} = \frac{1-\cos^2\theta}{\sin\theta\,(1+\cos\theta)} = \frac{\sin^2\theta}{\sin\theta\,(1+\cos\theta)},$$

so we also get:

$$\tan\tfrac{1}{2}\theta = \frac{\sin\theta}{1+\cos\theta} \tag{3.3.13}$$

Taking reciprocals in Equations 3.3.12 and 3.3.13 gives:

$$\cot\tfrac{1}{2}\theta = \frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta} \tag{3.3.14}$$

Since secant is the reciprocal of cosine, taking the reciprocal of Equation 3.3.7 for $\cos^2\frac{1}{2}\theta$ gives us

$$\sec^2\tfrac{1}{2}\theta = \frac{2}{1+\cos\theta} = \frac{2}{1+\cos\theta}\cdot\frac{\sec\theta}{\sec\theta} = \frac{2\sec\theta}{\sec\theta+1}.$$

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3.4: Other Identities

Though the identities in this section fall under the category of "other", they are perhaps (along with $\cos^2\theta+\sin^2\theta=1$) the most widely used identities in practice. It is very common to encounter terms such as $\sin A+\sin B$ or $\sin A\cos B$ in calculations, so we will now derive identities for those expressions. First, we have what are often called the *product-to-sum formulas*:

$$\sin A\cos B = \tfrac{1}{2}(\sin(A+B)+\sin(A-B)) \tag{3.4.1}$$
$$\cos A\sin B = \tfrac{1}{2}(\sin(A+B)-\sin(A-B)) \tag{3.4.2}$$
$$\cos A\cos B = \tfrac{1}{2}(\cos(A+B)+\cos(A-B)) \tag{3.4.3}$$
$$\sin A\sin B = -\tfrac{1}{2}(\cos(A+B)-\cos(A-B)) \tag{3.4.4}$$

We will prove the first formula; the proofs of the others are similar (see Exercises 1-3). We see that

$$\begin{aligned}\sin(A+B)+\sin(A-B) &= (\sin A\cos B+\cancel{\cos A\sin B})+(\sin A\cos B-\cancel{\cos A\sin B})\\ &= 2\sin A\cos B,\end{aligned}$$

so Equation 3.4.1 follows upon dividing both sides by 2. Notice how in each of the above identities a product (e.g. $\sin A\cos B$) of trigonometric functions is shown to be equivalent to a sum (e.g. $\frac{1}{2}(\sin(A+B)+\sin(A-B))$) of such functions. We can go in the opposite direction, with the *sum-to-product formulas*:

$$\sin A+\sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \tag{3.4.5}$$
$$\sin A-\sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \tag{3.4.6}$$
$$\cos A+\cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \tag{3.4.7}$$
$$\cos A-\cos B = -2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \tag{3.4.8}$$

These formulas are just the product-to-sum formulas rewritten by using some clever substitutions: let $x=\frac{1}{2}(A+B)$ and $y=\frac{1}{2}(A-B)$. Then $x+y=A$ and $x-y=B$. For example, to derive Equation 3.4.7, make the above substitutions in Equation 3.4.3 to get

$$\begin{aligned}\cos A+\cos B &= \cos(x+y)+\cos(x-y)\\ &= 2\cdot\tfrac{1}{2}(\cos(x+y)+\cos(x-y))\\ &= 2\cos x\cos y \qquad \text{(by Equation 3.4.3)}\\ &= 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B).\end{aligned}$$

The proofs of the other sum-to-product formulas are similar (see Exercises 4-6).

We are now in a position to prove Mollweide's equations, which we introduced in Section 2.3: For any triangle $\triangle ABC$,

$$\frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C} \quad\text{and}\quad \frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}.$$

First, since $C=2\cdot\frac{1}{2}C$, by the double-angle formula we have $\sin C=2\sin\frac{1}{2}C\cos\frac{1}{2}C$. Thus,

$$\begin{aligned}
\frac{a-b}{c} &= \frac{a}{c} - \frac{b}{c} = \frac{\sin A}{\sin C} - \frac{\sin B}{\sin C} \quad \text{(by the Law of Sines)} \\
&= \frac{\sin A - \sin B}{\sin C} = \frac{\sin A - \sin B}{2 \sin \frac{1}{2}C \cos \frac{1}{2}C} \\
&= \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}C \cos \frac{1}{2}C} \quad \text{(by Equation 3.4.6)} \\
&= \frac{\cos \frac{1}{2}(180^\circ - C) \sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}C \cos \frac{1}{2}C} \quad \text{(since } A+B = 180^\circ - C\text{)} \\
&= \frac{\cos (90^\circ - \frac{1}{2}C) \sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}C \cos \frac{1}{2}C} \\
&= \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}C} \quad \text{(since } \cos (90^\circ - \tfrac{1}{2}C) = \sin \tfrac{1}{2}C\text{)}.
\end{aligned}$$

This proves the first equation. The proof of the other equation is similar (see Exercise 7).

Using Mollweide's equations, we can prove the Law of Tangents: For any triangle $\triangle ABC$,

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}, \quad \frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)}, \quad \frac{c-a}{c+a} = \frac{\tan \frac{1}{2}(C-A)}{\tan \frac{1}{2}(C+A)}.$$

We need only prove the first equation; the other two are obtained by cycling through the letters. We see that

$$\begin{aligned}
\frac{a-b}{a+b} &= \frac{\frac{a-b}{c}}{\frac{a+b}{c}} = \frac{\frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}C}}{\frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C}} \quad \text{(by Mollweide's equations)} \\
&= \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A-B)} \cdot \frac{\sin \frac{1}{2}C}{\cos \frac{1}{2}C} \\
&= \tan \tfrac{1}{2}(A-B) \cdot \tan \tfrac{1}{2}C = \tan \tfrac{1}{2}(A-B) \cdot \tan (90^\circ - \tfrac{1}{2}(A+B)) \quad \text{(since } C = 180^\circ - (A+B)\text{)} \\
&= \tan \tfrac{1}{2}(A-B) \cdot \cot \tfrac{1}{2}(A+B) \quad \text{(since } \tan (90^\circ - \tfrac{1}{2}(A+B)) = \cot \tfrac{1}{2}(A+B)\text{, see Section 1.5)} \\
&= \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}. \quad \textbf{QED}
\end{aligned}$$

Example 3.18

For any triangle $\triangle ABC$, show that

$$\cos A + \cos B + \cos C = 1 + 4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C.$$

Solution

Since $\cos (A+B+C) = \cos 180^\circ = -1$, we can rewrite the left side as

$$\begin{aligned}
\cos A + \cos B + \cos C &= 1 + (\cos (A+B+C) + \cos C) + (\cos A + \cos B), \text{ so by Equation 3.4.7} \\
&= 1 + 2 \cos \tfrac{1}{2}(A+B+2C) \cos \tfrac{1}{2}(A+B) + 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B) \\
&= 1 + 2 \cos \tfrac{1}{2}(A+B) \left(\cos \tfrac{1}{2}(A+B+2C) + \cos \tfrac{1}{2}(A-B)\right), \text{ so} \\
&= 1 + 2 \cos \tfrac{1}{2}(A+B) \cdot 2 \cos \tfrac{1}{2}(A+C) \cos \tfrac{1}{2}(B+C) \text{ by Equation 3.4.7,}
\end{aligned}$$

since $\frac{1}{2}\left(\frac{1}{2}(A+B+2C)+\frac{1}{2}(A-B)\right)=\frac{1}{2}(A+C)$ and $\frac{1}{2}\left(\frac{1}{2}(A+B+2C)-\frac{1}{2}(A-B)\right)=\frac{1}{2}(B+C)$. Thus,}

$$\begin{aligned}\cos A+\cos B+\cos C &= 1+4\cos\left(90^\circ-\tfrac{1}{2}C\right)\cos\left(90^\circ-\tfrac{1}{2}B\right)\cos\left(90^\circ-\tfrac{1}{2}A\right)\\ &= 1+4\sin\tfrac{1}{2}C\sin\tfrac{1}{2}B\sin\tfrac{1}{2}A\ ,\ \text{so rearranging the order gives}\\ &= 1+4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\ .\end{aligned}$$

Let $u=\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$. Apply Equation 3.4.4 to the first two terms in u to get

$$u=-\tfrac{1}{2}\left(\cos\tfrac{1}{2}(A+B)-\cos\tfrac{1}{2}(A-B)\right)\sin\tfrac{1}{2}C=\tfrac{1}{2}\left(\cos\tfrac{1}{2}(A-B)-\cos\tfrac{1}{2}(A+B)\right)\cos\tfrac{1}{2}(A+B)\,,$$

since $\sin\frac{1}{2}C=\cos\frac{1}{2}(A+B)$, as we saw in Example 3.18. Multiply both sides by 2 to get

$$\cos^2\tfrac{1}{2}(A+B)-\cos\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B)+2u=0\,,$$

after rearranging the terms. Notice that the expression above is a quadratic equation in the term $\cos\frac{1}{2}(A+B)$. So by the quadratic formula,

$$\cos\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(A-B)\pm\sqrt{\cos^2\frac{1}{2}(A-B)-4(1)(2u)}}{2}\ ,$$

which has a real solution only if the quantity inside the square root is nonnegative. But we know that $\cos\frac{1}{2}(A+B)$ is a real number (and, hence, a solution exists), so we must have

$$\cos^2\tfrac{1}{2}(A-B)-8u\geq 0\quad\Rightarrow\quad u\leq\tfrac{1}{8}\cos^2\tfrac{1}{2}(A-B)\leq\tfrac{1}{8}\quad\Rightarrow\quad\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\leq\tfrac{1}{8}\,.$$

Example 3.20

For any triangle $\triangle ABC$, show that $1<\cos A+\cos B+\cos C\leq\frac{3}{2}$.

Solution

Since $0^\circ<A,\ B,\ C<180^\circ$, the sines of $\frac{1}{2}A$, $\frac{1}{2}B$, and $\frac{1}{2}C$ are all positive, so

$$\cos A+\cos B+\cos C=1+4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C>1$$

by Example 3.18. Also, by Examples 3.18 and 3.19 we have

$$\cos A+\cos B+\cos C=1+4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\leq 1+4\cdot\tfrac{1}{8}=\tfrac{3}{2}\,.$$

Hence, $1<\cos A+\cos B+\cos C\leq\frac{3}{2}$.

can be written as:

$$t_{1\,2\,p}=\frac{2\cos\theta_1\sin\theta_2}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}\ .$$

Solution

Multiply the top and bottom of $t_{1\,2\,p}$ by $\sin\theta_1\sin\theta_2$ to get:

$$
\begin{aligned}
t_{1\,2\,p} &= \frac{2\,n_1\,\cos\,\theta_1}{n_2\,\cos\,\theta_1 + n_1\,\cos\,\theta_2} \cdot \frac{\sin\,\theta_1\,\sin\,\theta_2}{\sin\,\theta_1\,\sin\,\theta_2} \\
&= \frac{2\,(n_1\,\sin\,\theta_1)\,\cos\,\theta_1\,\sin\,\theta_2}{(n_2\,\sin\,\theta_2)\,\sin\,\theta_1\,\cos\,\theta_1 + (n_1\,\sin\,\theta_1)\,\sin\,\theta_2\,\cos\,\theta_2} \\
&= \frac{2\,\cos\,\theta_1\,\sin\,\theta_2}{\sin\,\theta_1\,\cos\,\theta_1 + \sin\,\theta_2\,\cos\,\theta_2} \qquad \text{(by Snell's law)} \\
&= \frac{2\,\cos\,\theta_1\,\sin\,\theta_2}{\frac{1}{2}\,(\sin\,2\,\theta_1 + \sin\,2\theta_2)} \qquad \text{(by the double-angle formula)} \\
&= \frac{2\,\cos\,\theta_1\,\sin\,\theta_2}{\frac{1}{2}\,(2\,\sin\,\frac{1}{2}(2\theta_1+2\theta_2)\,\cos\,\frac{1}{2}(2\theta_1-2\theta_2))} \qquad \text{(by formula 3.4.5)} \\
&= \frac{2\,\cos\,\theta_1\,\sin\,\theta_2}{\sin\,(\theta_1+\theta_2)\,\cos\,(\theta_1-\theta_2)}
\end{aligned}
$$

$i(t)$ are given by

$$
\begin{aligned}
v(t) &= V_m\,\cos\,\omega t\,, \\
i(t) &= I_m\,\cos\,(\omega t+\phi)\,,
\end{aligned}
$$

for some constants V_m, I_m, ω, and ϕ. Show that the instantaneous power can be written as

$$
p(t) = \tfrac{1}{2}\,V_m\,I_m\,\cos\,\phi + \tfrac{1}{2}\,V_m\,I_m\,\cos\,(2\omega t+\phi)\,.
$$

Solution

By definition of $p(t)$, we have

$$
\begin{aligned}
p(t) &= V_m\,I_m\,\cos\,\omega t\,\cos\,(\omega t+\phi) \\
&= V_m\,I_m\,\cdot\,\tfrac{1}{2}(\cos\,(2\omega t+\phi) + \cos\,(-\phi)) && \text{(by Equation 3.4.3)} \\
&= \tfrac{1}{2}\,V_m\,I_m\,\cos\,\phi + \tfrac{1}{2}\,V_m\,I_m\,\cos\,(2\omega t+\phi) && \text{(since } \cos\,(-\phi) = \cos\,\phi\text{)}\,.
\end{aligned}
$$

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3.E: Identities (Exercises)

3.1 Exercises

3.1.1 We showed that $\sin\theta = \pm\sqrt{1-\cos^2\theta}$ for all θ. Give an example of an angle θ such that $\sin\theta = -\sqrt{1-\cos^2\theta}$.

3.1.2 We showed that $\cos\theta = \pm\sqrt{1-\sin^2\theta}$ for all θ. Give an example of an angle θ such that $\cos\theta = -\sqrt{1-\sin^2\theta}$.

3.1.3 Suppose that you are given a system of two equations of the following form:

$$A\cos\phi = B\nu_1 - B\nu_2\cos\theta$$
$$A\sin\phi = B\nu_2\sin\theta.$$

Show that $A^2 = B^2\left(\nu_1^2+\nu_2^2-2\nu_1\nu_2\cos\theta\right)$.

For Exercises 4-16, prove the given identity.

3.1.4 $\cos\theta\tan\theta = \sin\theta$

3.1.5 $\sin\theta\cot\theta = \cos\theta$

3.1.6 $\dfrac{\tan\theta}{\cot\theta} = \tan^2\theta$

3.1.7 $\dfrac{\csc\theta}{\sin\theta} = \csc^2\theta$

3.1.8 $\dfrac{\cos^2\theta}{1+\sin\theta} = 1-\sin\theta$

3.1.9 $\dfrac{1-2\cos^2\theta}{\sin\theta\cos\theta} = \tan\theta-\cot\theta$

3.1.10 $\sin^4\theta-\cos^4\theta = \sin^2\theta-\cos^2\theta$

3.1.11 $\cos^4\theta-\sin^4\theta = 1-2\sin^2\theta$

3.1.12 $\dfrac{1-\tan\theta}{1+\tan\theta} = \dfrac{\cot\theta-1}{\cot\theta+1}$

3.1.13 $\dfrac{\tan\theta+\tan\phi}{\cot\theta+\cot\phi} = \tan\theta\tan\phi$

3.1.14 $\dfrac{\sin^2\theta}{1-\sin^2\theta} = \tan^2\theta$

3.1.15 $\dfrac{1-\tan^2\theta}{1-\cot^2\theta} = 1-\sec^2\theta$

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3.1.16 $\sin\ \theta\ =\ \pm\ \dfrac{\tan\ \theta}{\sqrt{1\ +\ \tan^2\ \theta}}$ (*Hint: Solve for* $\sin^2\theta$ *in Exercise 14.*)

3.1.17 Sometimes identities can be proved by geometrical methods. For example, to prove the identity in Exercise 16, draw an acute angle θ in QI and pick the point $(1, y)$ on its terminal side, as in Figure 3.1.2. What must y equal? Use that to prove the identity for acute θ. Explain the adjustment(s) you would need to make in Figure 3.1.2 to prove the identity for θ in the other quadrants. Does the identity hold if θ is on either axis?

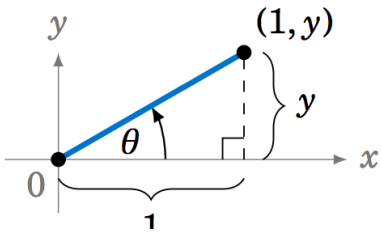


Figure 3.1.2

3.1.18 Similar to Exercise 16 , find an expression for $\cos\ \theta$ solely in terms of $\tan\ \theta$.

3.1.19 Find an expression for $\tan\ \theta$ solely in terms of $\sin\ \theta$, and one solely in terms of $\cos\ \theta$.

3.1.20 Suppose that a point with coordinates $(x, y) = (a\ (\cos\ \psi\ -\ \epsilon), a\sqrt{1-\epsilon^2}\ \sin\ \psi)$ is a distance $r > 0$ from the origin, where $a > 0$ and $0 < \epsilon < 1$. Use $r^2 = x^2 + y^2$ to show that $r = a\ (1\ -\ \epsilon\ \cos\ \psi)$.\\(Note: These coordinates arise in the study of elliptical orbits of planets.)

3.1.21 Show that each trigonometric function can be put in terms of the sine function.

3.2 Exercises

3.2.1 Verify the addition formulas 3.12 and 3.13 for $A = B = 0^\circ$.

For Exercises 2 and 3, find the exact values of $\sin\ (A+B)$, $\cos\ (A+B)$, and $\tan\ (A+B)$.

3.2.2 $\sin\ A = \frac{8}{17}$, $\cos\ A = \frac{15}{17}$, $\sin\ B = \frac{24}{25}$, $\cos\ B = \frac{7}{25}$

3.2.3 $\sin\ A = \frac{40}{41}$, $\cos\ A = \frac{9}{41}$, $\sin\ B = \frac{20}{29}$, $\cos\ B = \frac{21}{29}$

3.2.4 Use $75^\circ = 45^\circ + 30^\circ$ to find the exact value of $\sin\ 75^\circ$.

3.2.5 Use $15^\circ = 45^\circ - 30^\circ$ to find the exact value of $\tan\ 15^\circ$.

3.2.6 Prove the identity $\sin\ \theta + \cos\ \theta = \sqrt{2}\ \sin\ (\theta + 45^\circ)$. Explain why this shows that

$$-\sqrt{2}\ \leq\ \sin\ \theta\ +\ \cos\ \theta\ \leq\ \sqrt{2}$$

for all angles θ. For which θ between 0° and 360° would $\sin\ \theta\ +\ \cos\ \theta$ be the largest?

For Exercises 7-14, prove the given identity.

3.2.7

$$\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$$

3.2.8 $$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan A \tan C - \tan A \tan B}$$

3.2.9 $$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$$

3.2.10 $$\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

3.2.11 $$\tan(\theta+45^\circ) = \frac{1 + \tan\theta}{1 - \tan\theta}$$

3.2.12 $$\frac{\cos(A+B)}{\sin A \cos B} = \cot A - \tan B$$

3.2.13 $$\cot A + \cot B = \frac{\sin(A+B)}{\sin A \sin B}$$

3.2.14 $$\frac{\sin(A-B)}{\sin(A+B)} = \frac{\cot B - \cot A}{\cot B + \cot A}$$

3.2.15 Generalize Exercise 6: For any a and b, $-\sqrt{a^2+b^2} \le a \sin\theta + b \cos\theta \le \sqrt{a^2+b^2}$ for all θ.

3.2.16 Continuing Example 3.12, use Snell's law to show that the *s-polarization transmission Fresnel coefficient*

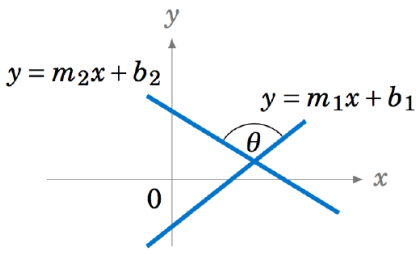
$$t_{1\,2\,s} = \frac{2\, n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \tag{3.22}$$

can be written as:

$$t_{1\,2\,s} = \frac{2 \cos\theta_1 \sin\theta_2}{\sin(\theta_2+\theta_1)}$$

3.2.17 Suppose that two lines with slopes m_1 and m_2, respectively, intersect at an angle θ and are not perpendicular (i.e. $\theta \neq 90^\circ$), as in the figure on the right. Show that

$$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| .$$



(*Hint: Use Example 1.26 from Section 1.5.*)

3.2.18 Use Exercise 17 to find the angle between the lines $y = 2x + 3$ and $y = -5x - 4$.

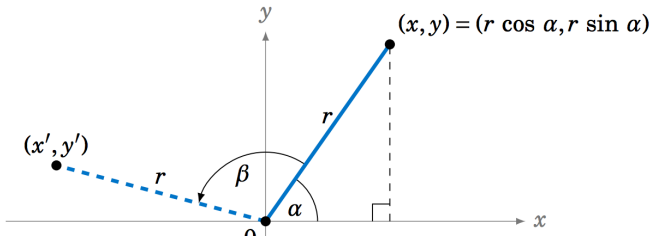
3.2.19 For any triangle $\triangle ABC$, show that $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$.
(Hint: Use Exercise 9 and $C = 180^\circ - (A+B)$.)

3.2.20 For any positive angles A, B, and C such that $A+B+C = 90^\circ$, show that

$$\tan A \tan B + \tan B \tan C + \tan C \tan A = 1 .$$

3.2.21 Prove the identity $\sin(A+B)\cos B - \cos(A+B)\sin B = \sin A$. Note that the right side depends only on A, while the left side depends on both A and B.

3.2.22 A line segment of length $r > 0$ from the origin to the point (x, y) makes an angle α with the positive x-axis, so that $(x, y) = (r\cos\alpha, r\sin\alpha)$, as in the figure below. What are the endpoint's new coordinates (x', y') after a counterclockwise rotation by an angle β ? Your answer should be in terms of r, α, and β.



3.3 Exercises

For Exercises 1-8, prove the given identity.

3.3.1 $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

3.3.2 $\tan\frac{1}{2}\theta = \csc\theta - \cot\theta$

3.3.3 $\dfrac{\sin 2\theta}{\sin\theta} - \dfrac{\cos 2\theta}{\cos\theta} = \sec\theta$

3.3.4 $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2$

3.3.5 $\tan 2\theta = \dfrac{2}{\cot\theta - \tan\theta}$

3.3.6 $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

3.3.7 $\tan^2\frac{1}{2}\theta = \dfrac{\tan\theta - \sin\theta}{\tan\theta + \sin\theta}$

3.3.8 $\dfrac{\cos^2\psi}{\cos^2\theta} = \dfrac{1 + \cos 2\psi}{1 + \cos 2\theta}$

3.3.9 Some trigonometry textbooks used to claim incorrectly that $\sin\theta + \cos\theta = \sqrt{1 + \sin 2\theta}$ was an identity. Give an example of a specific angle θ that would make that equation false. Is $\sin\theta + \cos\theta = \pm\sqrt{1 + \sin 2\theta}$ an identity? Justify your answer.

3.3.10 Fill out the rest of the table below for the angles $0^\circ < \theta < 720^\circ$ in increments of 90°, showing θ, $\frac{1}{2}\theta$, and the signs ($+$ or $-$) of $\sin\ \theta$ and $\tan\ \frac{1}{2}\theta$.

θ	$\frac{1}{2}\theta$	$\sin\theta$	$\tan\frac{1}{2}\theta$
$0^\circ - 90^\circ$	$0^\circ - 45^\circ$	$+$	$+$
$90^\circ - 180^\circ$	$45^\circ - 90^\circ$		
$180^\circ - 270^\circ$	$90^\circ - 135^\circ$		
$270^\circ - 360^\circ$	$135^\circ - 180^\circ$		

θ	$\frac{1}{2}\theta$	$\sin\theta$	$\tan\frac{1}{2}\theta$
$360^\circ - 450^\circ$	$180^\circ - 225^\circ$		
$450^\circ - 540^\circ$	$225^\circ - 270^\circ$		
$540^\circ - 630^\circ$	$270^\circ - 315^\circ$		
$630^\circ - 720^\circ$	$315^\circ - 360^\circ$		

3.3.11 In general, what is the largest value that $\sin\ \theta\ \cos\ \theta$ can take? Justify your answer.

For Exercises 12-17, prove the given identity for any right triangle $\triangle ABC$ with $C = 90^\circ$.

3.3.12 $\sin\ (A-B)\ =\ \cos\ 2B$

3.3.13 $\cos\ (A-B)\ =\ \sin\ 2A$

3.3.14 $\sin\ 2A\ =\ \dfrac{2\ ab}{c^2}$

3.3.15 $\cos\ 2A\ =\ \dfrac{b^2-a^2}{c^2}$

3.3.16 $\tan\ 2A\ =\ \dfrac{2\ ab}{b^2-a^2}$

3.3.17 $\tan\ \frac{1}{2}A\ =\ \dfrac{c-b}{a}\ =\ \dfrac{a}{c+b}$

3.3.18 Continuing Exercise 20 from Section 3.1, it can be shown that

$$\begin{aligned} r\ (1\ -\ \cos\ \theta)\ &=\ a\ (1\ +\ \epsilon)\ (1\ -\ \cos\ \psi)\ ,\ \text{and} \\ r\ (1\ +\ \cos\ \theta)\ &=\ a\ (1\ -\ \epsilon)\ (1\ +\ \cos\ \psi)\ , \end{aligned}$$

where θ and ψ are always in the same quadrant. Show that $\tan\ \frac{1}{2}\theta\ =\ \sqrt{\frac{1\ +\ \epsilon}{1\ -\ \epsilon}}\ \tan\ \frac{1}{2}\psi$.

3.4 Exercises

3.4.1 Prove formula 3.38.

3.4.2 Prove formula 3.39.

3.4.3 Prove formula 3.40.

3.4.4 Prove formula 3.41.

3.4.5 Prove formula 3.42.

3.4.6 Prove formula 3.44.

3.4.7 Prove Mollweide's second equation: For any triangle $\triangle ABC$, $\dfrac{a+b}{c}\ =\ \dfrac{\cos\ \frac{1}{2}(A-B)}{\sin\ \frac{1}{2}C}$.

3.4.8 Continuing Example 3.21, use Snell's law to show that the *p-polarization reflection Fresnel coefficient*

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$$r_{1\,2\,p} = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \tag{3.46}$$

can be written as:

$$r_{1\,2\,p} = \frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)} \tag{3.E.1}$$

3.4.9 There is a more general form for the instantaneous power $p(t) = v(t)\, i(t)$ in an electrical circuit than the one in Example 3.22. The voltage $v(t)$ and current $i(t)$ can be given by

$$\begin{aligned} v(t) &= V_m \cos(\omega t + \theta)\,, \\ i(t) &= I_m \cos(\omega t + \phi)\,, \end{aligned}$$

where θ is called the *phase angle*. Show that $p(t)$ can be written as

$$p(t) = \tfrac{1}{2} V_m I_m \cos(\theta - \phi) + \tfrac{1}{2} V_m I_m \cos(2\omega t + \theta + \phi)\,. \tag{3.E.2}$$

For Exercises 10-15, prove the given identity or inequality for any triangle $\triangle ABC$.

3.4.10 $\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C$ (*Hint: Mimic Example 3.18 using* $(\sin A + \sin B) + (\sin C - \sin(A+B+C))$.)

3.4.11 $\cos A + \cos(B - C) = 2 \sin B \sin C$

3.4.12 $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$ (*Hints: Group* $\sin 2B$ *and* $\sin 2C$ *together, use the double-angle formula for* $\sin 2A$*, use Exercise 11.*)

3.4.13 $\dfrac{a-b}{a+b} = \dfrac{\sin A - \sin B}{\sin A + \sin B}$

3.4.14 $\cos \tfrac{1}{2}A = \sqrt{\dfrac{s(s-a)}{bc}}$ and $\sin \tfrac{1}{2}A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, \;where $s = \tfrac{1}{2}(a+b+c)$$ (*Hint: Use the Law of Cosines to show that* $2bc(1 + \cos A) = 4s(s-a)$.)

3.4.15 $\tfrac{1}{2}(\sin A + \sin B) \leq \sin \tfrac{1}{2}(A+B)$ (*Hint: Show that* $\sin \tfrac{1}{2}(A+B) - \tfrac{1}{2}(\sin A + \sin B) \geq 0$.)

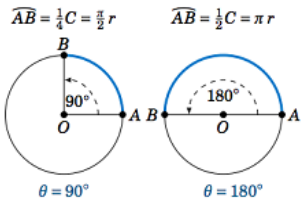
3.4.16 In Example 3.20, which angles A, B, C give the maximum value of $\cos A + \cos B + \cos C$?

CHAPTER OVERVIEW

4: RADIAN MEASURE

4.1: RADIANS AND DEGREES

So far we have been using degrees as our unit of measurement for angles. However, there is another way of measuring angles that is often more convenient. The idea is simple: associate a central angle of a circle with the arc that it intercepts.



4.2: ARC LENGTH

So suppose that we have a circle of radius r and we place a central angle with radian measure 1 on top of another central angle with radian measure 1, as in Figure 4.2.1(a). Clearly, the combined central angle of the two angles has radian measure 1+1 = 2, and the combined arc length is r + r = 2r.

4.3: AREA OF A SECTOR

In geometry you learned that the area of a circle of radius r is πr^2 . We will now learn how to find the area of a sector of a circle. A sector is the region bounded by a central angle and its intercepted arc, such as the shaded region in Figure 4.3.1.

4.4: CIRCULAR MOTION- LINEAR AND ANGULAR SPEED

So suppose that an object moves along a circle of radius r, traveling a distance s over a period of time t, as in Figure 4.4.1. Then it makes sense to define the (average) linear speed v of the object as: $v = \frac{s}{t}$. Let θ be the angle swept out by the object in that period of time. Then we define the (average) angular speed ω of the object as: $\omega = \frac{\theta}{t}$.

4.E: RADIAN MEASURE (EXERCISES)

These are homework exercises to accompany Corral's "Elementary Trigonometry" Textmap. This is a text on elementary trigonometry, designed for students who have completed courses in high-school algebra and geometry. Though designed for college students, it could also be used in high schools. The traditional topics are covered, but a more geometrical approach is taken than usual. Also, some numerical methods (e.g. the secant method for solving trigonometric equations) are discussed.

4.1: Radians and Degrees

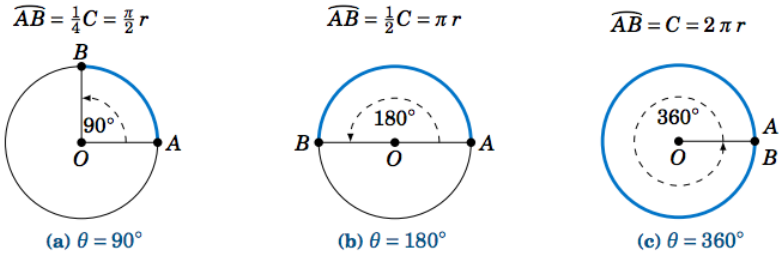


Figure 4.1.1 Angle θ and intercepted arc $\widehat{AB}$ on circle of circumference $C = 2\pi r$

In Figure 4.1.1 we see that a central angle of $90°$ cuts off an arc of length $\frac{\pi}{2}\,r$, a central angle of $180°$ cuts off an arc of length $\pi\,r$, and a central angle of $360°$ cuts off an arc of length $2\pi\,r$, which is the same as the circumference of the circle. So associating the central angle with its intercepted arc, we could say, for example, that

$$360° \quad \text{"equals''} \quad 2\pi\,r \quad (\text{or } 2\pi \text{ 'radiuses'}).$$

The radius r was arbitrary, but the 2π in front of it stays the same. So instead of using the awkward "radiuses" or "radii", we use the term **radians**:

$$\boxed{360° \;=\; 2\pi \text{ radians}} \tag{4.1.1}$$

The above relation gives us any easy way to convert between degrees and radians:

$$\textbf{Degrees to radians:} \quad x \text{ degrees} \;=\; \left(\frac{\pi}{180} \cdot x\right) \text{ radians} \tag{4.1.2}$$

$$\textbf{Radians to degrees:} \quad x \text{ radians} \;=\; \left(\frac{180}{\pi} \cdot x\right) \text{ degrees} \tag{4.1.3}$$

Equation 4.1.2 follows by dividing both sides of Equation 4.1.1 by 360, so that $1° = \frac{2\pi}{360} = \frac{\pi}{180}$ radians, then multiplying both sides by x. Equation 4.1.3 is similarly derived by dividing both sides of Equation 4.1.1 by 2π then multiplying both sides by x.

The statement $\theta = 2\pi$ radians is usually abbreviated as $\theta = 2\pi$ rad, or just $\theta = 2\pi$ when it is clear that we are using radians. When an angle is given as some multiple of π, you can assume that the units being used are radians.

Example 4.1

Convert $18°$ to radians.

Solution

Using the conversion Equation 4.1.2 for degrees to radians, we get

$$18° \;=\; \frac{\pi}{180} \cdot 18 \;=\; \boxed{\frac{\pi}{10} \text{ rad}}.$$

Example 4.2

Convert $\frac{\pi}{9}$ radians to degrees.

Solution

Using the conversion Equation 4.1.3 for radians to degrees, we get

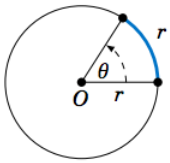
$$\frac{\pi}{9} \text{ rad} \;=\; \frac{180}{\pi} \cdot \frac{\pi}{9} \;=\; \boxed{20°}.$$

Table 4.1 Commonly used angles in radians

Degrees	Radians	Degrees	Radians	Degrees	Radians	Degrees	Radians
0°	0	90°	$\frac{\pi}{2}$	180°	π	270°	$\frac{3\pi}{2}$
30°	$\frac{\pi}{6}$	120°	$\frac{2\pi}{3}$	210°	$\frac{7\pi}{6}$	300°	$\frac{5\pi}{3}$
45°	$\frac{\pi}{4}$	135°	$\frac{3\pi}{4}$	225°	$\frac{5\pi}{4}$	315°	$\frac{7\pi}{4}$
60°	$\frac{\pi}{3}$	150°	$\frac{5\pi}{6}$	240°	$\frac{4\pi}{3}$	330°	$\frac{11\pi}{6}$

Table 4.1 shows the conversion between degrees and radians for some common angles. Using the conversion Equation 4.1.3 for radians to degrees, we see that

$$1 \text{ radian } = \frac{180}{\pi} \text{ degrees } \approx 57.3^\circ .$$



$\theta = 1$ radian

Figure 4.1.2

Formally, a radian is defined as the central angle in a circle of radius r which intercepts an arc of length r, as in Figure 4.1.2. This definition does not depend on the choice of r (imagine resizing Figure 4.1.2).

One reason why radians are used is that the scale is smaller than for degrees. One revolution in radians is $2\pi \approx 6.283185307$ which is much smaller than 360, the number of degrees in one revolution. The smaller scale makes the graphs of trigonometric functions (which we will discuss in Chapter 5) have similar scales for the horizontal and vertical axes. Another reason is that often in physical applications the variables being used are in terms of arc length, which makes radians a natural choice.

The default mode in most scientific calculators is to use degrees for entering angles. On many calculators there is a button labeled $\boxed{DRG}$ for switching between degree mode (D), radian mode (R), and *gradian* mode (G). On some graphing calculators, such as the the TI-83, there is a $\boxed{MODE}$ button for changing between degrees and radians. Make sure that your calculator is in the correct angle mode *before* entering angles, or your answers will likely be way off. For example,

$$\begin{aligned} \sin 4^\circ &= 0.0698 , \\ \sin (4 \text{ rad}) &= -0.7568 , \end{aligned}$$

so the values are not only off in magnitude, but do not even have the same sign. Using your calculator's $\boxed{\sin^{-1}}$, $\boxed{\cos^{-1}}$, and $\boxed{\tan^{-1}}$ buttons in radian mode will of course give you the angle as a decimal, not an expression in terms of π.

You should also be aware that the math functions in many computer programming languages use radians, so you would have to write your own angle conversions.

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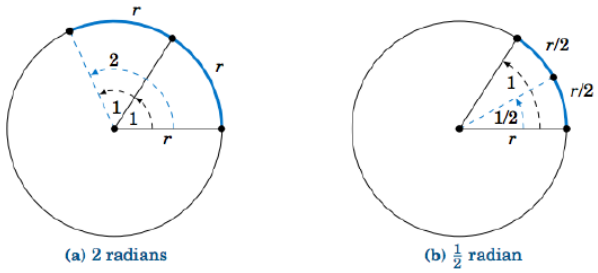
4.2: Arc Length

In Section 4.1 we saw that one revolution has a radian measure of 2π rad. Note that 2π is the ratio of the circumference (i.e. total arc length) C of a circle to its radius r:

$$\text{Radian measure of 1 revolution} = 2\pi = \frac{2\pi r}{r} = \frac{C}{r} = \frac{\text{total arc length}}{\text{radius}}$$

Clearly, that ratio is independent of r. In general, the radian measure of an angle is the ratio of the arc length cut off by the corresponding central angle in a circle to the radius of the circle, independent of the radius.



(a) 2 radians (b) $\frac{1}{2}$ radian

Figure 4.2.1 Radian measure and arc length

Now suppose that we cut the angle with radian measure 1 in half, as in Figure 4.2.1(b). Clearly, this cuts the arc length r in half as well. Thus, we see that

$$\begin{aligned} \text{Angle} &= 1 \text{ radian} &\Rightarrow\quad \text{arc length} &= r\,, \\ \text{Angle} &= 2 \text{ radians} &\Rightarrow\quad \text{arc length} &= 2r\,, \\ \text{Angle} &= \tfrac{1}{2} \text{ radian} &\Rightarrow\quad \text{arc length} &= \tfrac{1}{2}r\,, \end{aligned}$$

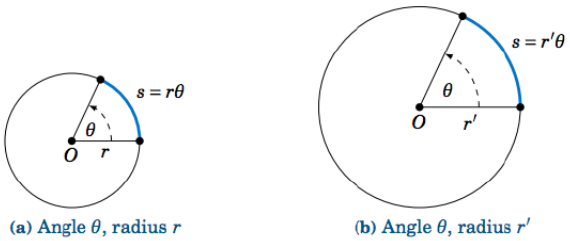
and in general, for any $\theta \geq 0$,

$$\text{Angle} = \theta \text{ radians} \quad\Rightarrow\quad \text{arc length} = \theta r\,,$$

so that

$$\theta = \frac{\text{arc length}}{\text{radius}}\,.$$

Intuitively, it is obvious that shrinking or magnifying a circle preserves the measure of a central angle even as the radius changes. The above discussion says more, namely that the ratio of the length s of an intercepted arc to the radius r is preserved, precisely because that ratio is the measure of the central angle in radians (see Figure 4.2.2).



(a) Angle θ, radius r (b) Angle θ, radius r'

Figure 4.2.2 Circles with the same central angle, different radii

We thus get a simple formula for the length of an arc:

In a circle of radius r, let s be the length of an arc intercepted by a central angle with radian measure $\theta \geq 0$. Then the arc length s is:

$$s = r\,\theta \tag{4.2.1}$$

$s = r\,\theta = (2)\,(1.2) = \boxed{2.4 \text{ cm}}$

. So first convert $\theta = 41^\circ$ to radians, then use $s = r\,\theta$:

$$\theta = 41^\circ = \frac{\pi}{180} \cdot 41 = 0.716 \text{ rad} \quad \Rightarrow \quad s = r\,\theta = (10)\,(0.716) = \boxed{7.16 \text{ ft}}$$

Note that since the arc length s and radius r are usually given in the same units, radian measure is really unitless, since you can think of the units canceling in the ratio $\frac{s}{r}$, which is just θ. This is another reason why radians are so widely used.

$\theta = 0.4 \text{ rad} = \dfrac{180}{\pi} \cdot 0.4 = \boxed{22.92^\circ}$

For central angles $\theta > 2\pi$ rad, i.e. $\theta > 360^\circ$, it may not be clear what is meant by the intercepted arc, since the angle is larger than one revolution and hence "wraps around" the circle more than once. We will take the approach that such an arc consists of the full circumference plus any additional arc length determined by the angle. In other words, Equation 4.2.1 is still valid for angles $\theta > 2\pi$ rad.

What about negative angles? In this case using $s = r\,\theta$ would mean that the arc length is negative, which violates the usual concept of length. So we will adopt the convention of only using nonnegative central angles when discussing arc length.

A rope is fastened to a wall in two places 8 ft apart at the same height. A cylindrical container with a radius of 2 ft is pushed away from the wall as far as it can go while being held in by the rope, as in Figure 4.2.3 which shows the top view. If the center of the container is 3 feet away from the point on the wall midway between the ends of the rope, what is the length L of the rope?

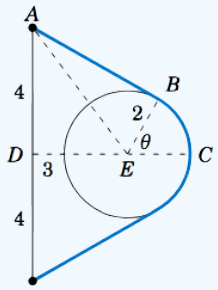


Figure 4.2.3

Solution:

We see that, by symmetry, the total length of the rope is $L = 2\,(AB + \overset{\frown}{BC})$. Also, notice that $\triangle ADE$ is a right triangle, so the hypotenuse has length $AE = \sqrt{DE^2 + DA^2} = \sqrt{3^2 + 4^2} = 5$ ft, by the Pythagorean Theorem. Now since $\overline{AB}$ is tangent to the circular container, we know that $\angle ABE$ is a right angle. So by the Pythagorean Theorem we have

$$AB = \sqrt{AE^2 - BE^2} = \sqrt{5^2 - 2^2} = \sqrt{21}\text{ ft.}$$

By Equation 4.2.1 the arc $\overset{\frown}{BC}$ has length $BE \cdot \theta$, where $\theta = \angle BEC$ is the supplement of $\angle AED + \angle AEB$. So since

$$\tan \angle AED = \frac{4}{3} \Rightarrow \angle AED = 53.1^\circ \quad \text{and} \quad \cos \angle AEB = \frac{BE}{AE} = \frac{2}{5} \Rightarrow \angle AEB = 66.4^\circ,$$

we have

$$\theta = \angle BEC = 180^\circ - (\angle AED + \angle AEB) = 180^\circ - (53.1^\circ + 66.4^\circ) = 60.5^\circ.$$

Converting to radians, we get $\theta = \frac{\pi}{180} \cdot 60.5 = 1.06$ rad. Thus,

$$L = 2(AB + \overset{\frown}{BC}) = 2(\sqrt{21} + BE \cdot \theta) = 2(\sqrt{21} + (2)(1.06)) = \boxed{13.4\text{ ft}}.$$

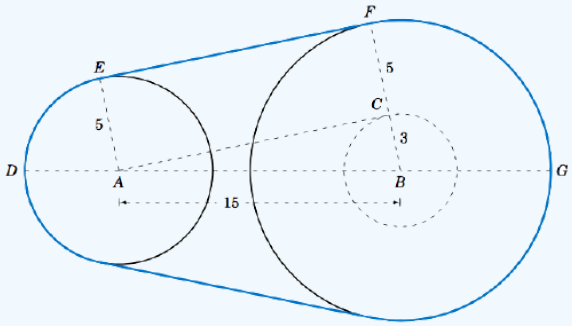


Figure 4.2.4 Belt pulleys with radii 5 cm and 8 cm

First, at the center B of the pulley with radius 8, draw a circle of radius 3, which is the difference in the radii of the two pulleys. Let C be the point where this circle intersects $\overline{BF}$. Then we know that the tangent line $\overline{AC}$ to this smaller circle is perpendicular to the line segment $\overline{BF}$. Thus, $\angle ACB$ is a right angle, and so the length of $\overline{AC}$ is

$$AC = \sqrt{AB^2 - BC^2} = \sqrt{15^2 - 3^2} = \sqrt{216} = 6\sqrt{6}$$

by the Pythagorean Theorem. Now since $\overline{AE} \perp \overline{EF}$ and $\overline{EF} \perp \overline{CF}$ and $\overline{CF} \perp \overline{AC}$, the quadrilateral $AEFC$ must be a rectangle. In particular, $EF = AC$, so $EF = 6\sqrt{6}$.

By Equation 4.2.1 we know that $\overset{\frown}{DE} = EA \cdot \angle DAE$ and $\overset{\frown}{FG} = BF \cdot \angle GBF$, where the angles are measured in radians. So thinking of angles in radians (using π rad $= 180^\circ$), we see from Figure 4.2.4 that

$$\angle DAE = \pi - \angle EAC - \angle BAC = \pi - \frac{\pi}{2} - \angle BAC = \frac{\pi}{2} - \angle BAC,$$

where

$$\sin \angle BAC = \frac{BC}{AB} = \frac{3}{15} = 0.2 \quad \Rightarrow \quad \angle BAC = 0.201\text{ rad.}$$

Thus, $\angle DAE = \frac{\pi}{2} - 0.201 = 1.37$ rad. So since $\overline{AE}$ and $\overline{BF}$ are parallel, we have $\angle ABC = \angle DAE = 1.37$ rad. Thus, $\angle GBF = \pi - \angle ABC = \pi - 1.37 = 1.77$ rad. Hence,

$$L = 2(\overset{\frown}{DE} + EF + \overset{\frown}{FG}) = 2(5(1.37) + 6\sqrt{6} + 8(1.77)) = \boxed{71.41\text{ cm}}.$$

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4.3: Area of a Sector

In geometry you learned that the area of a circle of radius r is πr^2. We will now learn how to find the area of a **sector** of a circle. A sector is the region bounded by a central angle and its intercepted arc, such as the shaded region in Figure 4.3.1.

Let θ be a central angle in a circle of radius r and let A be the area of its sector. Similar to arc length, the ratio of A to the area of the entire circle is the same as the ratio of θ to one revolution. In other words, again using radian measure,

$$\frac{\text{area of sector}}{\text{area of entire circle}} = \frac{\text{sector angle}}{\text{one revolution}} \quad \Rightarrow \quad \frac{A}{\pi\, r^2} = \frac{\theta}{2\pi}\,.$$

Solving for A in the above equation, we get the following formula:

In a circle of radius r, the area A of the sector inside a central angle θ is

$$A = \tfrac{1}{2}\, r^2\, \theta\,, \tag{4.3.1}$$

where θ is measured in radians.

Solution

Using $\theta = \frac{\pi}{5}$ and $r = 4$ in Equation 4.3.1, the area A of the sector is

$$A = \tfrac{1}{2}\, r^2\, \theta = \tfrac{1}{2}\,(4)^2 \cdot \tfrac{\pi}{5} = \boxed{\tfrac{8\pi}{5}\ \text{cm}^2}\,.$$

Example 4.9

Find the area of a sector whose angle is 117° in a circle of radius 3.5 m.

Solution:

As with arc length, we have to make sure that the angle is measured in radians or else the answer will be way off. So converting $\theta = 117^\circ$ to radians and using $r = 3.5$ in Equation 4.3.1 for the area A of the sector, we get

$$\theta = 117^\circ = \frac{\pi}{180} \cdot 117 = 2.042\ \text{rad} \quad \Rightarrow \quad A = \tfrac{1}{2}\, r^2\, \theta = \tfrac{1}{2}\,(3.5)^2\,(2.042) = \boxed{12.51\ \text{m}^2}\,.$$

For a sector whose angle is θ in a circle of radius r, the length of the arc cut off by that angle is $s = r\,\theta$. Thus, by Equation 4.3.1 the area A of the sector can be written as:

$$\boxed{A = \tfrac{1}{2}\, rs} \tag{4.3.2}$$

Note: The central angle θ that intercepts an arc is sometimes called the angle **subtended** by the arc.

Solution

Using $s = 6$ and $r = 9$ in Equation 4.3.2 for the area A, we get

$$A = \tfrac{1}{2}\, rs = \ = \tfrac{1}{2}\,(9)\,(6) = \boxed{27\ \text{cm}^2}\,.$$

Note that the angle subtended by the arc is $\theta = \frac{s}{r} = \frac{2}{3}$ rad.

Example 4.11

Find the area K inside the belt pulley system from Example 4.7 in Section 4.2.

Solution:

Recall that the belt pulleys have radii of 5 cm and 8 cm, and their centers are 15 cm apart. We showed in Example 4.7 that $EF = AC = 6\sqrt{6}$, $\angle DAE = 1.37$ rad, and $\angle GBF = 1.77$ rad. We see from Figure 4.3.2 that, by symmetry, the total area K enclosed by the belt is twice the area above the line $\overline{DG}$, that is,

$$\begin{aligned} K \;=\; 2\,(&(\text{Area of sector } DAE) \;+\; (\text{Area of rectangle } AEFC) \\ &+\; (\text{Area of triangle } \triangle ABC) \;+\; (\text{Area of sector } GBF))\,. \end{aligned}$$

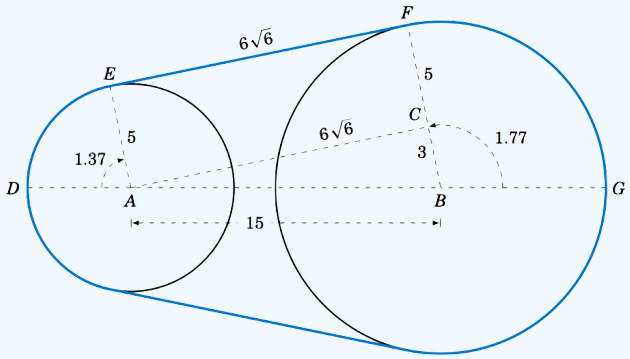

Figure 4.3.2 Belt pulleys with radii 5 cm and 8 cm

Since $AEFC$ is a rectangle with sides 5 and $6\sqrt{6}$, its area is $30\sqrt{6}$. And since $\triangle ABC$ is a right triangle whose legs have lengths 3 and $6\sqrt{6}$, its area is $\frac{1}{2}(3)(6\sqrt{6}) = 9\sqrt{6}$. Thus, using Equation 4.3.1 for the areas of sectors DAE and GBF, we have

$$\begin{aligned} K \;&=\; 2\,((\text{Area of sector } DAE) \;+\; 30\sqrt{6} \;+\; 9\sqrt{6} \;+\; (\text{Area of sector } GBF)) \\ &=\; 2\,\left(\tfrac{1}{2}(5)^2(1.37) \;+\; 30\sqrt{6} \;+\; 9\sqrt{6} \;+\; \tfrac{1}{2}(8)^2(1.77)\right) \\ &=\; \boxed{338.59 \text{ cm}^2}\,. \end{aligned}$$

So far we have dealt with the area cut off by a central angle. How would you find the area of a region cut off by an inscribed angle, such as the shaded region in Figure 4.3.3? In this picture, the center of the circle is inside the inscribed angle, and the lengths a and b of the two chords are given, as is the radius r of the circle. Drawing line segments from the center of the circle to the endpoints of the chords indicates how to solve this problem: add up the areas of the two triangles and the sector formed by the central angle. The areas and angles of the two triangles can be determined (since all three sides are known) using methods from Chapter 2. Also, recall (Theorem 2.4 in Section 2.5) that a central angle has twice the measure of any inscribed angle which intercepts the same arc. In the exercises you will be asked to solve problems like this (including the cases where the center of the circle is outside or on the inscribed angle).

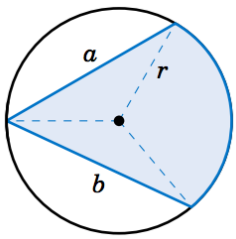

Figure 4.3.3

Another type of region we can consider is a **segment** of a circle, which is the region between a chord and the arc it cuts off. In Figure 4.3.4 the segment formed by the chord $\overline{AB}$ is the shaded region between the arc $\overset{\frown}{AB}$ and the triangle $\triangle OAB$. By Equation 2.23 in Section 2.4 for the area of a triangle given two sides and their included angle, we know that

$$\text{area of } \triangle OAB = \tfrac{1}{2}(r)(r)\sin\theta = \tfrac{1}{2}r^2\sin\theta\,.$$

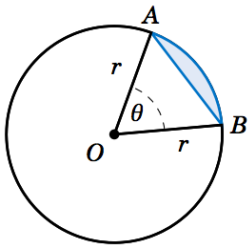

Figure 4.3.4

Thus, since the area K of the segment is the area of the sector AOB minus the area of the triangle $\triangle OAB$, we have

$$\text{area } K \text{ of segment } AB = \tfrac{1}{2}r^2\,\theta - \tfrac{1}{2}r^2\sin\theta = \boxed{\tfrac{1}{2}r^2(\theta - \sin\theta)}\,. \tag{4.3.3}$$

Note that as a consequence of Equation 4.3.3 we must have $\theta > \sin\theta$ for $0 < \theta \le \pi$ (measured in radians), since the area of a segment is positive for those angles.

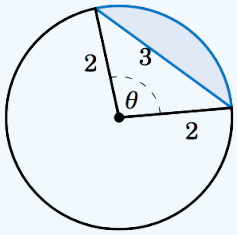

Figure 4.3.5

Solution

Figure 4.3.5 shows the segment formed by a chord of length 3 in a circle of radius $r = 2$. We can use the Law of Cosines to find the subtended central angle θ:

$$\cos\theta = \frac{2^2 + 2^2 - 3^2}{2\,(2)\,(2)} = -0.125 \quad \Rightarrow \quad \theta = 1.696 \text{ rad}$$

Thus, by Equation 4.3.3 the area K of the segment is:

$$K = \tfrac{1}{2}r^2(\theta - \sin\theta) = \tfrac{1}{2}(2)^2(1.696 - \sin 1.696) = \boxed{1.408}$$

Example 4.13

The centers of two circles are 7 cm apart, with one circle having a radius of 5 cm and the other a radius of 4 cm. Find the area K of their intersection.

Solution:

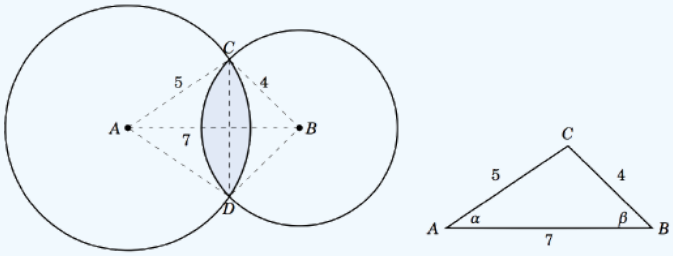

Figure 4.3.6

By symmetry, we see that $\angle BAC = \frac{1}{2}\angle CAD$ and $\angle ABC = \frac{1}{2}\angle CBD$. So let $\alpha = \angle BAC$ and $\beta = \angle ABC$, as in Figure 4.3.6(b). By the Law of Cosines, we have

$$\cos\,\alpha = \frac{7^2+5^2-4^2}{2\,(7)\,(5)} = 0.8286 \quad \Rightarrow \quad \alpha = 0.594 \text{ rad} \quad \Rightarrow \quad \angle CAD = 2\,(0.594) = 1.188 \text{ rad}$$

$$\cos\,\beta = \frac{7^2+4^2-5^2}{2\,(7)\,(4)} = 0.7143 \quad \Rightarrow \quad \beta = 0.775 \text{ rad} \quad \Rightarrow \quad \angle CBD = 2\,(0.775) = 1.550 \text{ rad}$$

Thus, the area K is

$$\begin{aligned} K &= (\text{Area of segment } CD \text{ in circle at } A) + (\text{Area of segment } CD \text{ in circle at } B) \\ &= \tfrac{1}{2}\,(5)^2\,(1.188 - \sin\,1.188) + \tfrac{1}{2}\,(4)^2\,(1.550 - \sin\,1.550) \\ &= \boxed{7.656 \text{ cm}^2}. \end{aligned}$$

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4.4: Circular Motion- Linear and Angular Speed

Radian measure and arc length can be applied to the study of *circular motion*. In physics the *average speed* of an object is defined as:

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time elapsed}}$$

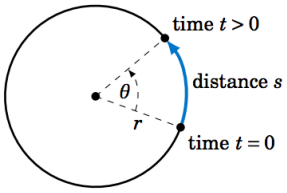

Figure 4.4.1

So suppose that an object moves along a circle of radius r, traveling a distance s over a period of time t, as in Figure 4.4.1. Then it makes sense to define the (average) **linear speed** ν of the object as:

$$\nu = \frac{s}{t} \tag{4.4.1}$$

Let θ be the angle swept out by the object in that period of time. Then we define the (average) **angular speed** ω of the object as:

$$\omega = \frac{\theta}{t} \tag{4.4.2}$$

Angular speed gives the rate at which the central angle swept out by the object changes as the object moves around the circle, and it is thus measured in radians per unit time. Linear speed is measured in distance units per unit time (e.g. feet per second). The word linear is used because straightening out the arc traveled by the object along the circle results in a line of the same length, so that the usual definition of speed as distance over time can be used. We will usually omit the word average when discussing linear and angular speed here.

Since the length s of the arc cut off by a central angle θ in a circle of radius r is $s = r\,\theta$, we see that

$$\nu = \frac{s}{t} = \frac{r\,\theta}{t} = \frac{\theta}{t} \cdot r\,,$$

so that we get the following relation between linear and angular speed:

$$\nu = \omega r \tag{4.4.3}$$

Example 4.14

An object sweeps out a central angle of $\frac{\pi}{3}$ radians in 0.5 seconds as it moves along a circle of radius 3 m. Find its linear and angular speed over that time period.

Solution:

Here we have $t = 0.5$ sec, $r = 3$ m, and $\theta = \frac{\pi}{3}$ rad. So the angular speed ω is

$$\omega = \frac{\theta}{t} = \frac{\frac{\pi}{3}\text{ rad}}{0.5\text{ sec}} \quad \Rightarrow \quad \boxed{\omega = \frac{2\pi}{3}\text{ rad/sec}}\,,$$

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and thus the linear speed ν is

$$\nu = \omega r = \left(\frac{2\pi}{3}\,\text{rad/sec}\right)(3\text{ m}) \quad \Rightarrow \quad \boxed{\nu = 2\pi\text{ m/sec}}\,.$$

Note that the units for ω are rad/sec and the units of ν are m/sec. Recall that radians are actually unitless, which is why in the Equation $\nu = \omega r$ the radian units disappear.

Example 4.15

An object travels a distance of 35 ft in 2.7 seconds as it moves along a circle of radius 2 ft. Find its linear and angular speed over that time period.

Solution:

Here we have $t = 2.7$ sec, $r = 2$ ft, and $s = 35$ ft. So the linear speed ν is

$$\nu = \frac{s}{t} = \frac{35\text{ feet}}{2.7\text{ sec}} \quad \Rightarrow \quad \boxed{\nu = 12.96\text{ ft/sec}}\,,$$

and thus the angular speed ω is given by

$$\nu = \omega r \quad \Rightarrow \quad 12.96\text{ ft/sec} = \omega(2\text{ ft}) \quad \Rightarrow \quad \boxed{\omega = 6.48\text{ rad/sec}}\,.$$

Example 4.16

An object moves at a constant linear speed of 10 m/sec around a circle of radius 4 m. How large of a central angle does it sweep out in 3.1 seconds?

Solution:

Here we have $t = 3.1$ sec, $\nu = 10$ m/sec, and $r = 4$ m. Thus, the angle θ is given by

$$s = r\theta \quad \Rightarrow \quad \theta = \frac{s}{r} = \frac{\nu t}{r} = \frac{(10\text{ m/sec})(3.1\text{ sec})}{4\text{ m}} = \boxed{7.75\text{ rad}}\,.$$

In many physical applications angular speed is given in *revolutions per minute*, abbreviated as *rpm*. To convert from rpm to, say, radians per second, notice that since there are 2π radians in one revolution and 60 seconds in one minute, we can convert N rpm to radians per second by "canceling the units" as follows:

$$N\text{ rpm} = N\,\frac{\cancel{\text{rev}}}{\cancel{\text{min}}}\cdot\frac{2\pi\text{ rad}}{1\ \cancel{\text{rev}}}\cdot\frac{1\ \cancel{\text{min}}}{60\text{ sec}} = \frac{N\cdot 2\pi}{60}\,\text{rad/sec}$$

This works because all we did was multiply by 1 twice. Converting to other units for angular speed works in a similar way. Going in the opposite direction, say, from rad/sec to rpm, gives:

$$N\,\text{rad/sec} = \frac{N\cdot 60}{2\pi}\,\text{rpm}$$

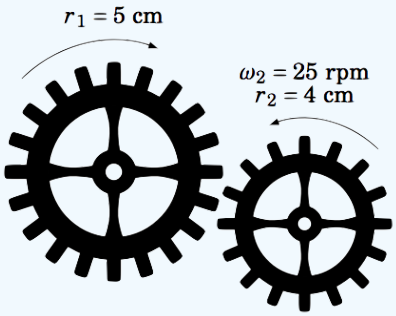

Figure 4.4.2

Solution

Imagine a particle on the outer radius of each gear. After the gears have rotated for a period of time $t > 0$, the circular displacement of each particle will be the same. In other words, $s_1 = s_2$, where s_1 and s_2 are the distances traveled by the particles on the gears with radii r_1 and r_2, respectively.

But $s_1 = \nu_1 t$ and $s_2 = \nu_2 t$, where ν_1 and ν_2 are the linear speeds of the gears with radii r_1 and r_2, respectively. Thus,

$$\nu_1 t = \nu_2 t \quad \Rightarrow \quad \nu_1 = \nu_2 ,$$

so by Equation 4.4.3 we get the fundamental relation between the two gears:

$$\boxed{\omega_1 r_1 = \omega_2 r_2} \tag{4.4.4}$$

Note that this holds for any two gears. So in our case, we have

$$\omega_1 (5) = (25)(4) \quad \Rightarrow \quad \boxed{\omega_1 = 20 \text{ rpm}} .$$

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4.E: Radian Measure (Exercises)

These are homework exercises to accompany Corral's "Elementary Trigonometry" Textmap. This is a text on elementary trigonometry, designed for students who have completed courses in high-school algebra and geometry. Though designed for college students, it could also be used in high schools. The traditional topics are covered, but a more geometrical approach is taken than usual. Also, some numerical methods (e.g. the secant method for solving trigonometric equations) are discussed.

4.1 Exercise

For Exercises 1-5, convert the given angle to radians.

4.1.1 4°

4.1.2 15°

4.1.3 130°

4.1.4 275°

4.1.5 -108°

For Exercises 6-10, convert the given angle to degrees.

4.1.6 4 rad

4.1.7 $\frac{\pi}{5}$ rad

4.1.8 $\frac{11\pi}{9}$ rad

4.1.9 $\frac{29\pi}{30}$ rad

4.1.10 35 rad

4.1.11 Put your calculator in radian mode and take the cosine of 0. Whatever the answer is, take its cosine. Then take the cosine of the new answer. Keep repeating this. On most calculators after about 50-60 iterations you should start to see the same answer repeating. What is that number? Try starting with a number different from 0. Do you get the same answer repeating after roughly the same number of iterations as before? Try the same procedure in degree mode, starting with 0°. Does the same thing happen? If so, does it take fewer iterations for the answer to start repeating than in radian mode, or more?

4.2 Exercise

For Exercises 1-4, find the length of the arc cut off by the given central angle θ in a circle of radius r.

4.2.1 $\theta = 0.8$ rad, $r = 12$ cm

4.2.2 $\theta = 171^\circ$, $r = 8$ m

4.2.3 $\theta = \pi$ rad, $r = 11$ in

4.2.4 A central angle in a circle of radius 2 cm cuts off an arc of length 4.6 cm. What is the measure of the angle in radians? What is the measure of the angle in degrees?

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4.2.5 The centers of two belt pulleys, with radii of 3 inches and 6 inches, respectively, are 13 inches apart. Find the total length L of the belt around the pulleys.

4.2.6 In Figure 4.2.5 one end of a 4 ft iron rod is attached to the center of a pulley with radius 0.5 ft. The other end is attached at a 40° angle to a wall, at a spot 6 ft above the lower end of a steel wire supporting a box. The other end of the wire comes out of the wall straight across from the top of the pulley. Find the length L of the wire from the wall to the box.

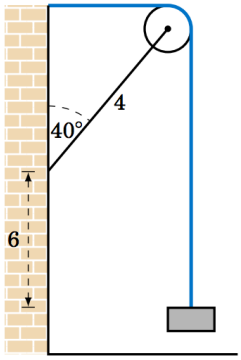

Figure 4.2.5 Exercise 6

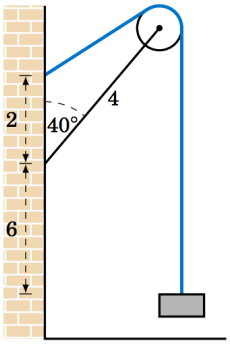

Figure 4.2.6 Exercise 7

4.2.7 Figure 4.2.6 shows the same setup as in Exercise 6 but now the wire comes out of the wall 2 ft above where the rod is attached. Find the length L of the wire from the wall to the box.

4.2.8 Find the total length L of the figure eight shape in Figure 4.2.7.

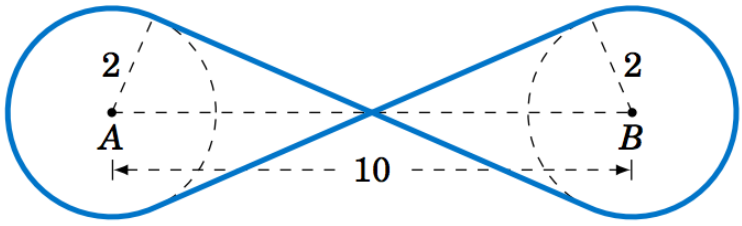

Figure 4.2.7

4.2.9 Repeat Exercise 8 but with the circle at A having a radius of 3 instead of 2. (*Hint: Draw a circle of radius* 5 *centered at* A*, then draw a tangent line to that circle from* B.)

4.2.10 Suppose that in Figure 4.2.7 the lines do not criss-cross but instead go straight across, as in a belt pulley system. Find the total length L of the resulting shape.

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4.2.11 Find the lengths of the two arcs cut off by a chord of length 3 in a circle of radius 2.

4.2.12 Find the perimeter of a *regular dodecagon* (i.e. a 12-sided polygon with sides of equal length) inscribed inside a circle of radius $\frac{1}{2}$. Compare it to the circumference of the circle.

4.3 Exercise

For Exercises 1-3, find the area of the sector for the given angle θ and radius r.

4.3.1 $\theta = 2.1$ rad, $r = 1.2$ cm

4.3.2 $\theta = \frac{3\pi}{7}$ rad, $r = 3.5$ ft

4.3.3 $\theta = 78^\circ$, $r = 6$ m

4.3.4 The centers of two belt pulleys, with radii of 3 cm and 6 cm, respectively, are 13 cm apart. Find the total area K enclosed by the belt.

4.3.5 In Exercise 4 suppose that both belt pulleys have the same radius of 6 cm. Find the total area K enclosed by the belt.

4.3.6 Find the area enclosed by the figure eight in Exercise 8 from Section 4.2.

For Exercises 7-9, find the area of the sector for the given radius r and arc length s.

4.3.7 $r = 5$ cm, $s = 2$ cm

4.3.8 $r = a$, $s = a$

4.3.9 $r = 1$ cm, $s = \pi$ cm

For Exercises 10-12, find the area of the segment formed by a chord of length a in a circle of radius r.

4.3.10 $a = 4$ cm, $r = 4$ cm

4.3.11 $a = 1$ cm, $r = 5$ cm

4.3.12 $a = 2$ cm, $r = 5$ cm

4.3.13 Find the area of the shaded region in Figure 4.3.7.

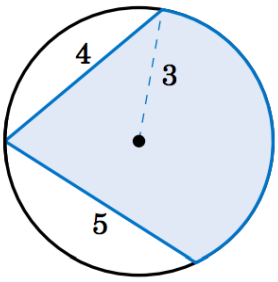

Figure 4.3.7 Exercise 13

4.3.14 Find the area of the shaded region in Figure 4.3.8. (*Hint: Draw two central angles.*)

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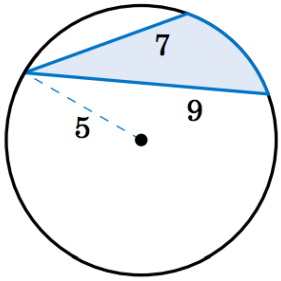

Figure 4.3.8 Exercise 14

4.3.15 Find the area of the shaded region in Figure 4.3.9.

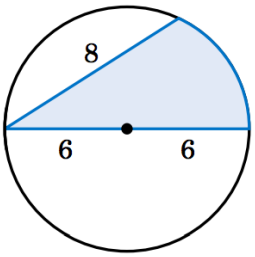

Figure 4.3.9 Exercise 15

4.3.16 The centers of two circles are 4 cm apart, with one circle having a radius of 3 cm and the other a radius of 2 cm. Find the area of their intersection.

4.3.17 Three circles with radii of 4 m, 2 m, and 1 m are externally tangent to each other. Find the area of the curved region between the circles, as in Figure 4.3.10. (*Hint: Connect the centers of the circles*.)

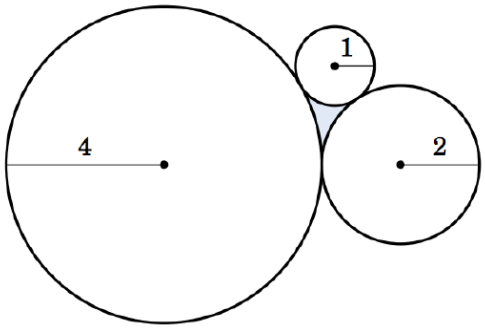

Figure 4.3.10 Exercise 17

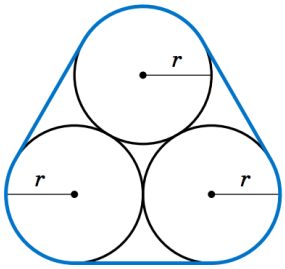

Figure 4.3.11 Exercise 18

4.3.18 Show that the total area enclosed by the loop around the three circles of radius r in Figure 4.3.11 is $(\pi + 6 + \sqrt{3})\, r^2$.

4.3.19 For a fixed central angle θ, how much does the area of its sector increase when the radius of the circle is doubled? How much does the length of its intercepted arc increase?

4.4 Exercise

For Exercises 1-6, assume that a particle moves along a circle of radius r for a period of time t. Given either the arc length s or the central angle θ swept out by the particle, find the linear and angular speed of the particle.

4.4.1 $r = 4$ m, $t = 2$ sec, $\theta = 3$ rad

4.4.2 $r = 8$ m, $t = 2$ sec, $\theta = 3$ rad

4.4.3 $r = 7$ m, $t = 3.2$ sec, $\theta = 172^\circ$

4.4.4 $r = 1$ m, $t = 1.6$ sec, $s = 3$ m

4.4.5 $r = 2$ m, $t = 1.6$ sec, $s = 6$ m

4.4.6 $r = 1.5$ ft, $t = 0.3$ sec, $s = 4$ in

4.4.7 An object moves at a constant linear speed of 6 m/sec around a circle of radius 3.2 m. How large of a central angle does it sweep out in 1.8 seconds?

4.4.8 Two interlocking gears have outer radii of 6 cm and 9 cm, respectively. If the smaller gear rotates at 40 rpm, how fast does the larger gear rotate?

4.4.9 Three interlocking gears have outer radii of 2 cm, 3 cm, and 4 cm, respectively. If the largest gear rotates at 16 rpm, how fast do the other gears rotate?

4.4.10 In Example 4.17, does equation 4.11 still hold if the radii r_1 and r_2 are replaced by the number of teeth N_1 and N_2, respectively, of the two gears as shown in Figure 4.4.2?

4.4.11 A 78 rpm music record has a diameter of 10 inches. What is the linear speed of a speck of dust on the outer edge of the record in inches per second?

4.4.12 The *centripetal acceleration* α of an object moving along a circle of radius r with a linear speed ν is defined as $\alpha = \frac{\nu^2}{r}$. Show that $\alpha = \omega^2 r$, where ω is the angular speed.

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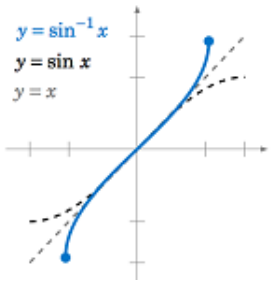
CHAPTER OVERVIEW

5: GRAPHING AND INVERSE FUNCTIONS



5.1: GRAPHING THE TRIGONOMETRIC FUNCTIONS

The trigonometric functions can be graphed just like any other function, as we will now show. In the graphs we will always use radians for the angle measure.

5.2: PROPERTIES OF GRAPHS OF TRIGONOMETRIC FUNCTIONS

We saw in Section 5.1 how the graphs of the trigonometric functions repeat every 2π radians. In this section we will discuss this and other properties of graphs, especially for the sinusoidal functions (sine and cosine).

5.3: INVERSE TRIGONOMETRIC FUNCTIONS

We have briefly mentioned the inverse trigonometric functions before, but we will now define those inverse functions and determine their graphs.

5.E: GRAPHING AND INVERSE FUNCTIONS (EXERCISES)

These are homework exercises to accompany Corral's "Elementary Trigonometry" Textmap. This is a text on elementary trigonometry, designed for students who have completed courses in high-school algebra and geometry. Though designed for college students, it could also be used in high schools. The traditional topics are covered, but a more geometrical approach is taken than usual. Also, some numerical methods (e.g. the secant method for solving trigonometric equations) are discussed.

5.1: Graphing the Trigonometric Functions

The first function we will graph is the sine function. We will describe a geometrical way to create the graph, using the *unit circle*. This is the circle of radius 1 in the xy-plane consisting of all points (x, y) which satisfy the equation $x^2 + y^2 = 1$.

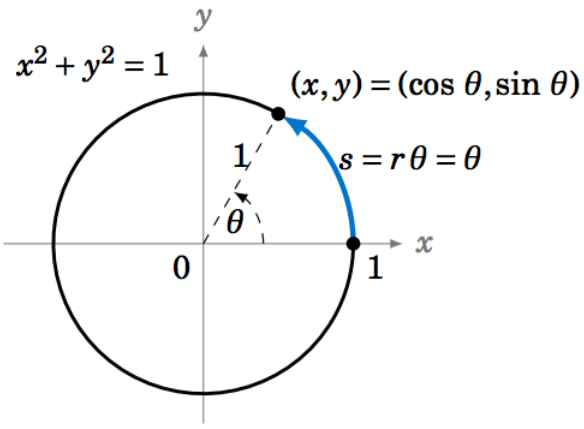


Figure 5.1.1

We see in Figure 5.1.1 that any point on the unit circle has coordinates $(x, y) = (\cos\ \theta, \sin\ \theta)$, where θ is the angle that the line segment from the
origin to (x, y) makes with the positive x-axis (by definition of sine and cosine). So as the point (x, y) goes around the circle, its y-coordinate is $\sin\ \theta$.

We thus get a correspondence between the y-coordinates of points on the unit circle and the values $f(\theta) = \sin\ \theta$, as shown by the horizontal lines from the unit circle to the graph of $f(\theta) = \sin\ \theta$ in Figure 5.1.2 for the angles $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$.

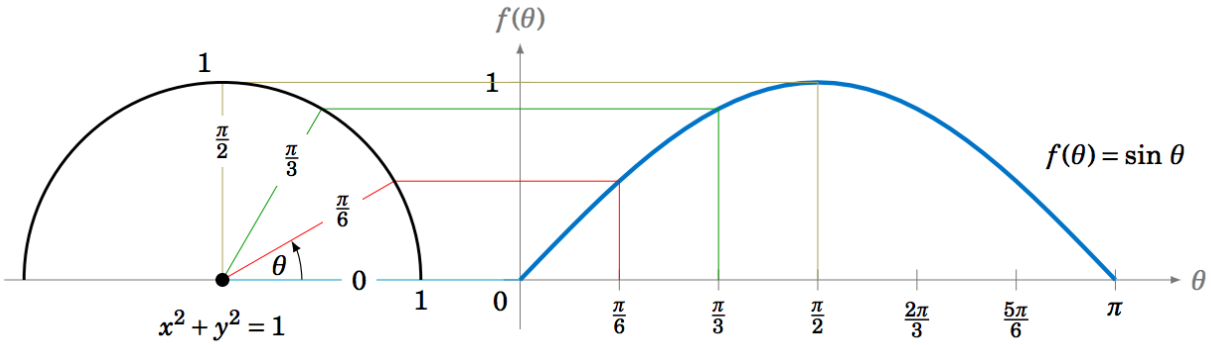


Figure 5.1.2 Graph of sine function based on y-coordinate of points on unit circle

We can extend the above picture to include angles from 0 to 2π radians, as in Figure 5.1.3. This illustrates what is sometimes called the *unit circle definition of the sine function.*

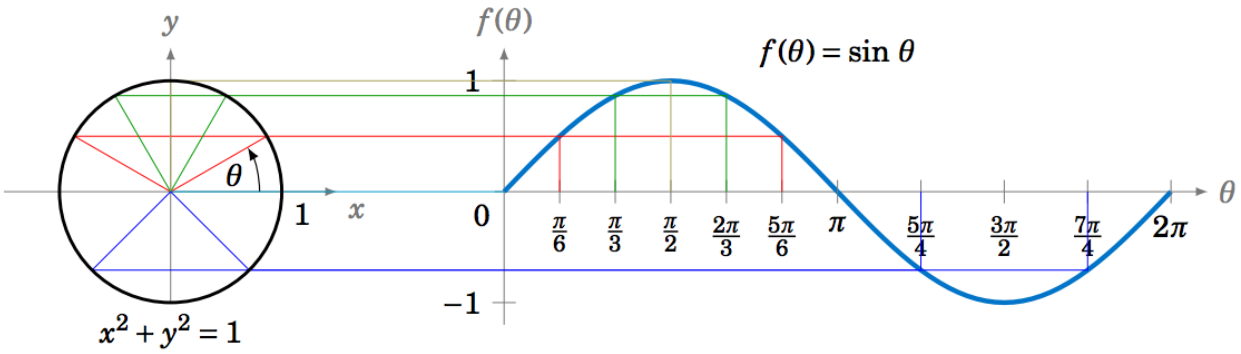


Figure 5.1.3 Unit circle definition of the sine function

Since the trigonometric functions repeat every 2π radians (360°), we get, for example, the following graph of the function $y = \sin\ x$ for x in the interval $[-2\pi, 2\pi]$:

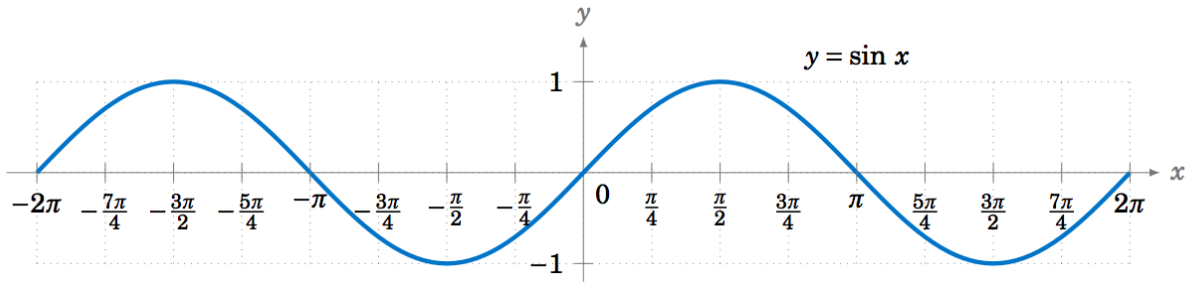


Figure 5.1.4 Graph of $y = \sin x$

To graph the cosine function, we could again use the unit circle idea (using the x-coordinate of a point that moves around the circle), but there is an easier way. Recall from Section 1.5 that $\cos\ x = \sin\ (x + 90^\circ)$ for all x. So $\cos\ 0^\circ$ has the same value as $\sin\ 90^\circ$, $\cos\ 90^\circ$ has the same value as $\sin\ 180^\circ$, $\cos\ 180^\circ$ has the same value as $\sin\ 270^\circ$, and so on. In other words, the graph of the cosine function is just the graph of the sine function shifted to the *left* by $90^\circ = \pi/2$ radians, as in Figure 5.1.5:

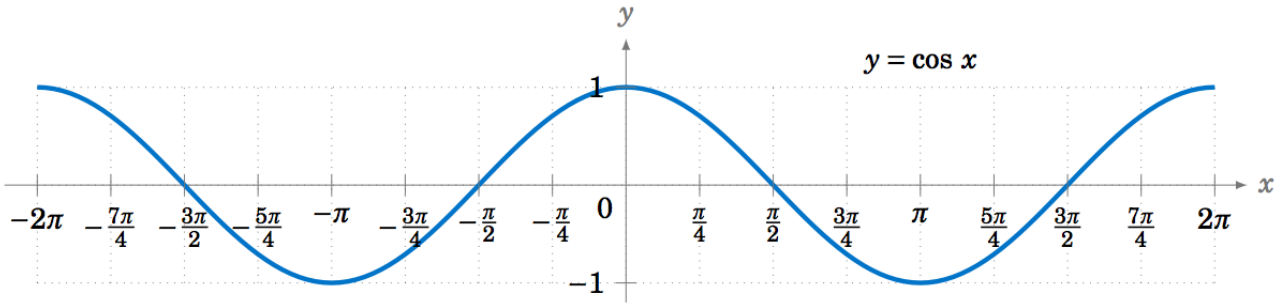


Figure 5.1.5 Graph of $y = \cos x$

To graph the tangent function, use $\tan\ x = \frac{\sin\ x}{\cos\ x}$ to get the following graph:

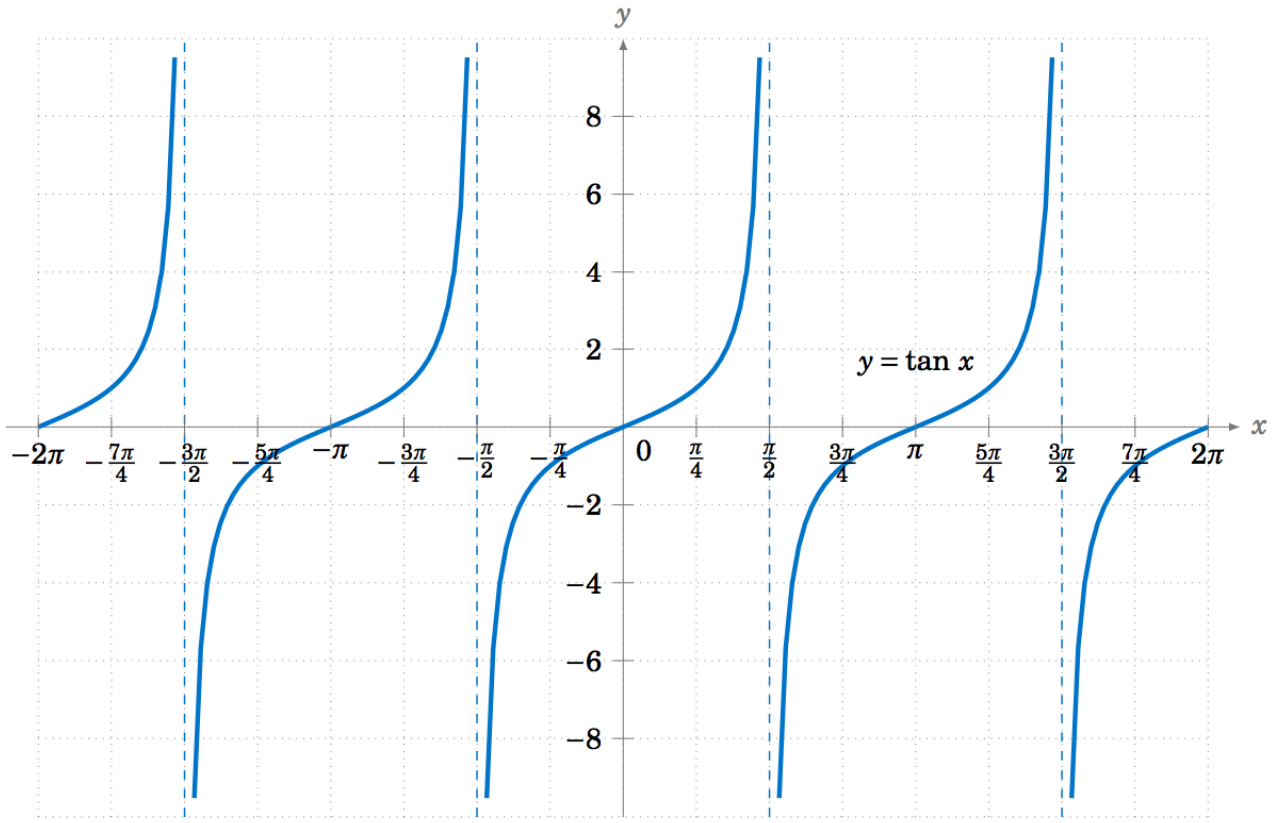


Figure 5.1.6 Graph of $y = \tan x$

Recall that the tangent is positive for angles in QI and QIII, and is negative in QII and QIV, and that is indeed what the graph in Figure 5.1.6 shows. We know that $\tan\ x$ is not defined when $\cos\ x = 0$, i.e. at odd multiples of $\frac{\pi}{2}$: $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, etc. We can figure out what happens *near* those angles by looking at the sine and cosine functions. For example, for x in QI near $\frac{\pi}{2}$, $\sin\ x$ and $\cos\ x$ are both positive, with $\sin\ x$ very close to 1 and $\cos\ x$ very close to 0, so the quotient $\tan\ x = \frac{\sin\ x}{\cos\ x}$ is a positive number that is very large. And the closer x gets to $\frac{\pi}{2}$, the larger $\tan\ x$ gets. Thus, $x = \frac{\pi}{2}$ is a *vertical asymptote* of the graph of $y = \tan\ x$.

Likewise, for x in QII very close to $\frac{\pi}{2}$, $\sin\ x$ is very close to 1 and $\cos\ x$ is negative and very close to 0, so the quotient $\tan\ x = \frac{\sin\ x}{\cos\ x}$ is a negative number that is very large, and it gets larger in the negative direction the closer x gets to $\frac{\pi}{2}$. The graph shows this. Similarly, we get vertical asymptotes at $x = -\frac{\pi}{2}$, $x = \frac{3\pi}{2}$, and $x = -\frac{3\pi}{2}$, as in Figure 5.1.6. Notice that the graph of the tangent function repeats every π radians, i.e. two times faster than the graphs of sine and cosine repeat.

The graphs of the remaining trigonometric functions can be determined by looking at the graphs of their reciprocal functions. For example, using $\csc\ x = \frac{1}{\sin\ x}$ we can just look at the graph of $y = \sin\ x$ and invert the values. We will get vertical asymptotes when $\sin\ x = 0$, namely at multiples of π: $x = 0, \pm\pi, \pm 2\pi$, etc. Figure 5.1.7 shows the graph of $y = \csc\ x$, with the graph of $y = \sin\ x$ (the dashed curve) for reference.

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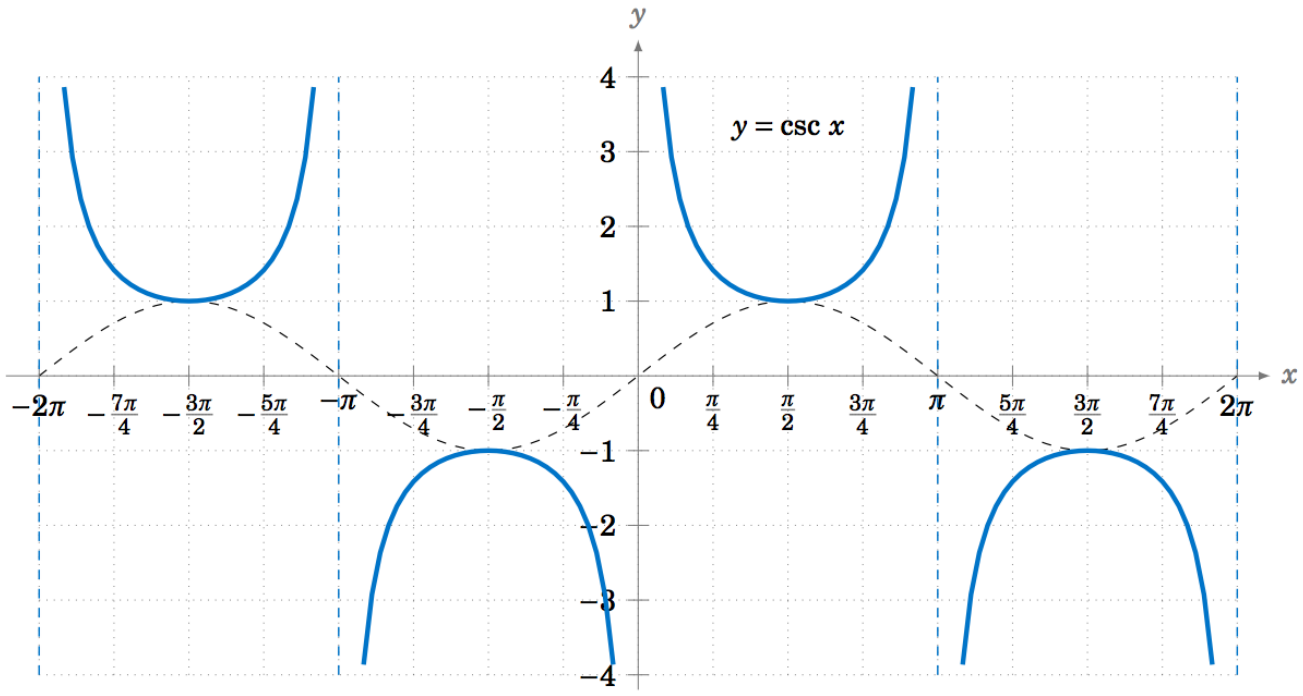


Figure 5.1.7 Graph of $y = \csc x$

Likewise, Figure 5.1.8 shows the graph of $y = \sec\ x$, with the graph of $y = \cos\ x$ (the dashed curve) for reference. Note the vertical asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$. Notice also that the graph is just the graph of the cosecant function shifted to the left by $\frac{\pi}{2}$ radians.

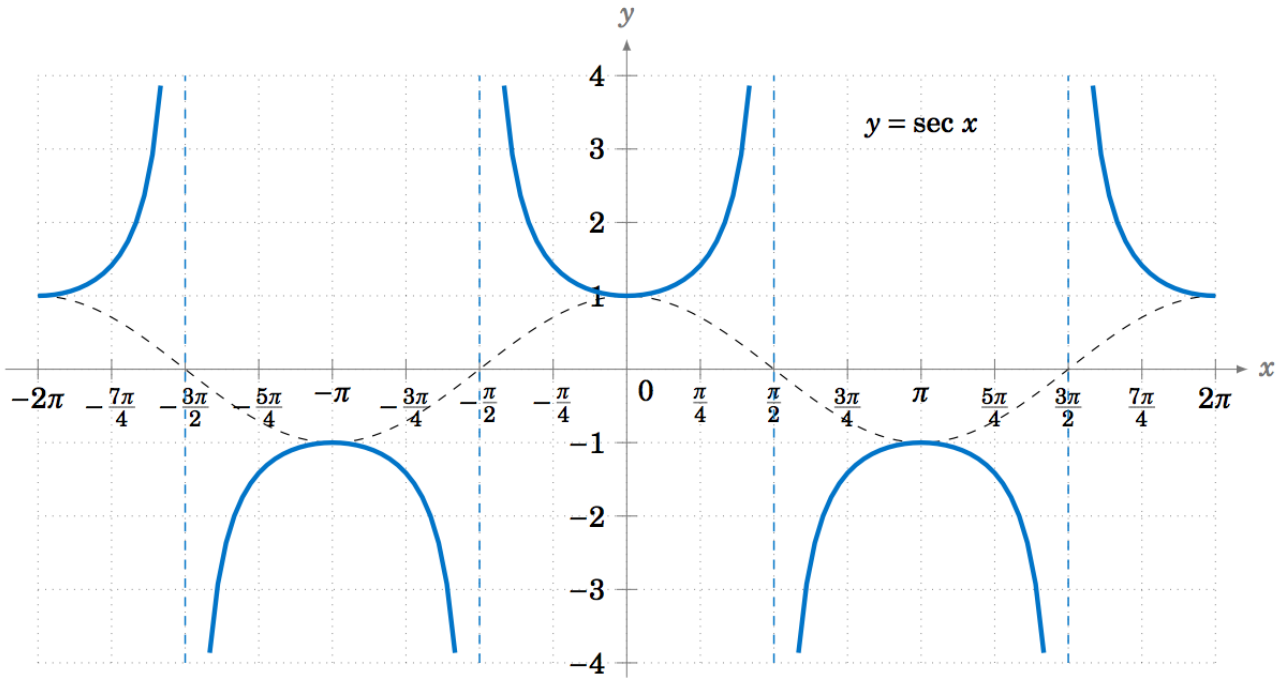


Figure 5.1.8 Graph of $y = \sec x$

The graph of $y = \cot\ x$ can also be determined by using $\cot\ x = \frac{1}{\tan\ x}$. Alternatively, we can use the relation $\cot\ x = -\tan\ (x + 90^\circ)$ from Section 1.5, so that the graph of the cotangent function is just the graph of the tangent function shifted to the left by $\frac{\pi}{2}$ radians and then reflected about the x-axis, as in Figure 5.1.9:

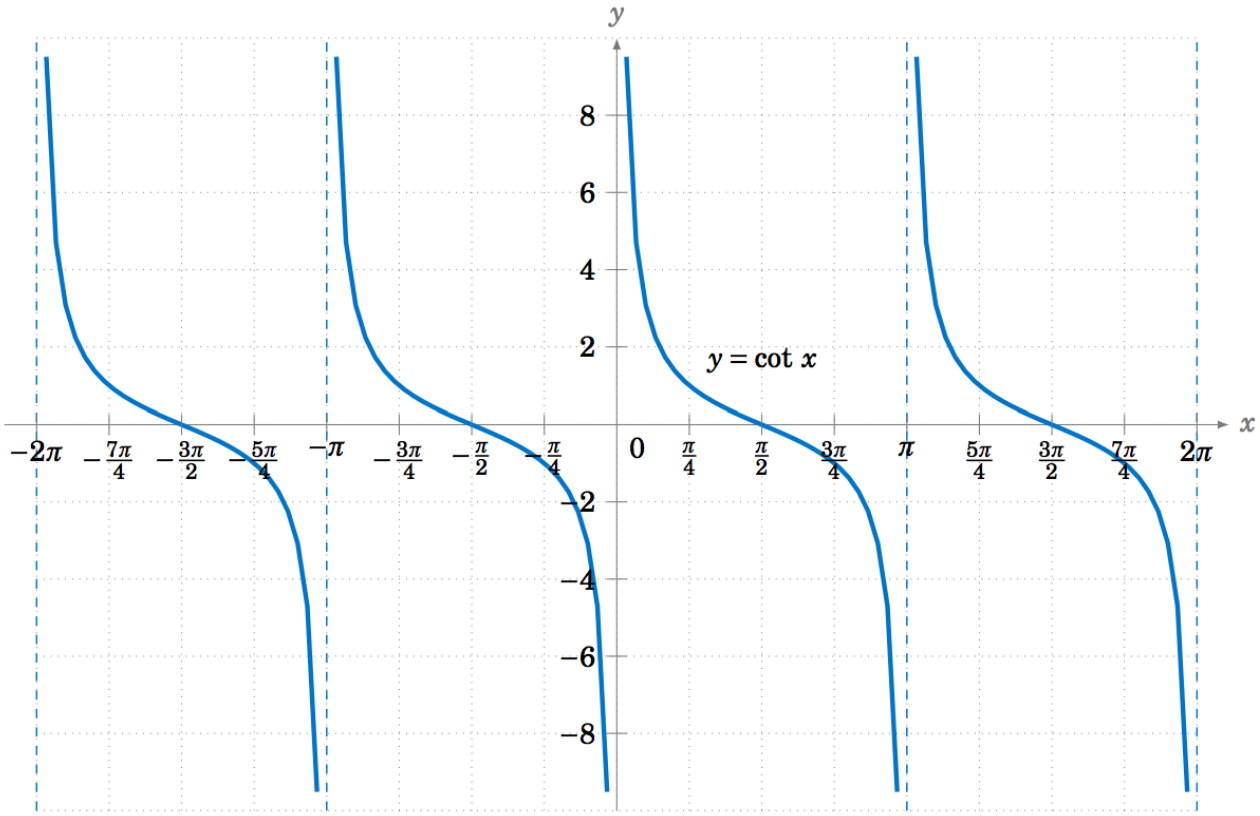


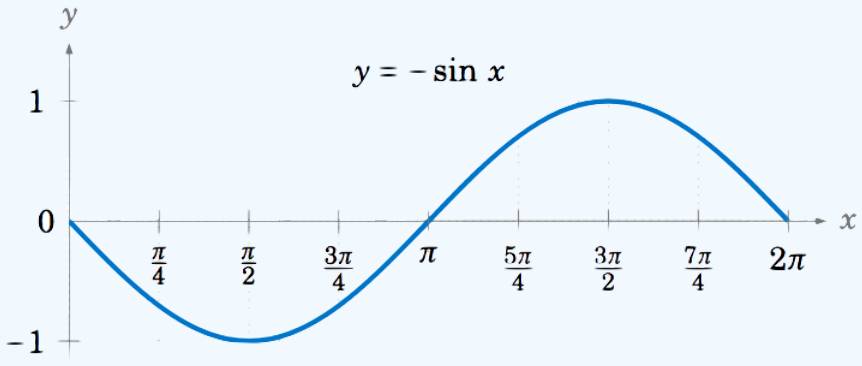
Figure 5.1.9 Graph of $y = \cot x$

Example 5.1.1

Draw the graph of $y = -\sin\ x$ for $0 \leq x \leq 2\pi$.

Solution:

Multiplying a function by -1 just reflects its graph around the x-axis. So reflecting the graph of $y = \sin\ x$ around the x-axis gives us the graph of $y = -\sin\ x$:



\noindent Note that this graph is the same as the graphs of $y = \sin\ (x \pm \pi)$ and $y = \cos\ (x + \frac{\pi}{2})$.

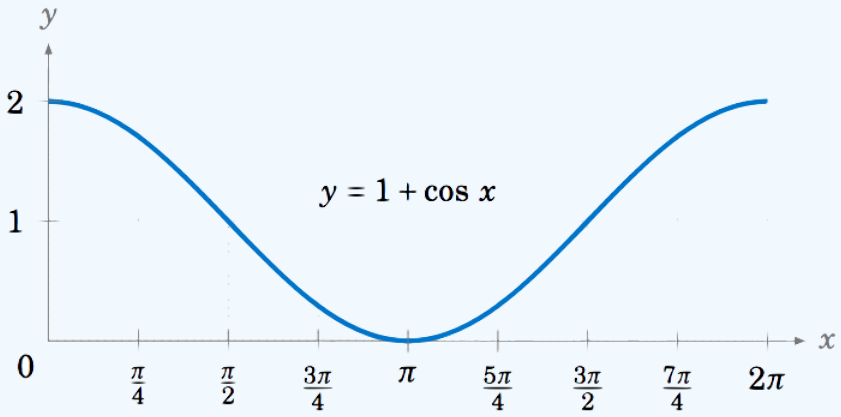
It is worthwhile to remember the general shapes of the graphs of the six trigonometric functions, especially for sine, cosine, and tangent. In particular, the graphs of the sine and cosine functions are called *sinusoidal* curves. Many phenomena in nature exhibit sinusoidal behavior, so recognizing the general shape is important.

Example 5.1.2

Draw the graph of $y = 1 + \cos\ x$ for $0 \leq x \leq 2\pi$.

Solution

Adding a constant to a function just moves its graph up or down by that amount, depending on whether the constant is positive or negative, respectively. So adding 1 to $\cos\ x$ moves the graph of $y = \cos\ x$ upward by 1, giving us the graph of $y = 1 + \cos\ x$:



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5.2: Properties of Graphs of Trigonometric Functions

We saw in Section 5.1 how the graphs of the trigonometric functions repeat every 2π radians. In this section we will discuss this and other properties of graphs, especially for the sinusoidal functions (sine and cosine).

First, recall that the **domain** of a function $f(x)$ is the set of all numbers x for which the function is defined. For example, the domain of $f(x) = \sin\ x$ is the set of all real numbers, whereas the domain of $f(x) = \tan\ x$ is the set of all real numbers except $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$. The **range** of a function $f(x)$ is the set of all values that $f(x)$ can take over its domain. For example, the range of $f(x) = \sin\ x$ is the set of all real numbers between -1 and 1 (i.e. the interval $[-1, 1]$), whereas the range of $f(x) = \tan\ x$ is the set of all real numbers, as we can see from their graphs.

A function $f(x)$ is **periodic** if there exists a number $p > 0$ such that $x + p$ is in the domain of $f(x)$ whenever x is, and if the following relation holds:

$$f(x+p) = f(x) \quad \text{for all } x \tag{5.2.1}$$

There could be many numbers p that satisfy the above requirements. If there is a smallest such number p, then we call that number the **period** of the function $f(x)$.

Example 5.2.1

The functions $\sin\ x$, $\cos\ x$, $\csc\ x$, and $\sec\ x$ all have the same period: 2π radians. We saw in Section 5.1 that the graphs of $y = \tan\ x$ and $y = \cot\ x$ repeat every 2π radians but they also repeat every π radians. Thus, the functions $\tan\ x$ and $\cot\ x$ have a period of π radians.

Example 5.2.2

What is the period of $f(x) = \sin\ 2x$?

Solution

The graph of $y = \sin\ 2x$ is shown in Figure 5.2.1, along with the graph of $y = \sin\ x$ for comparison, over the interval $[0, 2\pi]$. Note that $\sin\ 2x$ ``goes twice as fast" as $\sin\ x$.

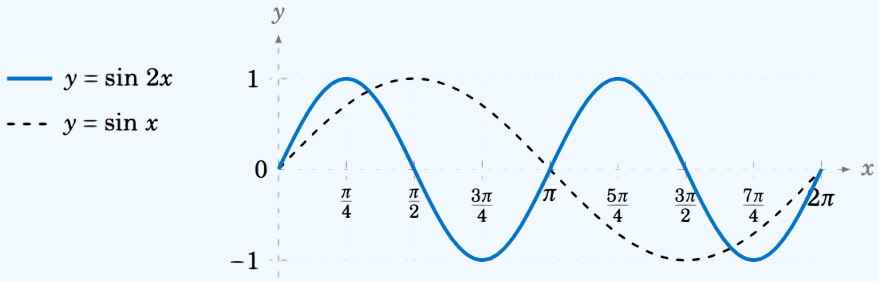

Figure 5.2.1: Graph of $y = \sin 2x$

For example, for x from 0 to $\frac{\pi}{2}$, $\sin\ x$ goes from 0 to 1, but $\sin\ 2x$ is able to go from 0 to 1 quicker, just over the interval $[0, \frac{\pi}{4}]$. While $\sin\ x$ takes a full 2π radians to go through an entire *cycle* (the largest part of the graph that does not repeat), $\sin\ 2x$ goes through an entire cycle in just π radians. So the period of $\sin\ 2x$ is π radians.

The above example made use of the graph of $\sin\ 2x$, but the period can be found analytically. Since $\sin\ x$ has period 2π, we know that $\sin\ (x + 2\pi) = \sin\ x$ for all x. Since $2x$ is a number for all x, this means in particular that $\sin\ (2x + 2\pi) = \sin\ 2x$ for all x. Now define $f(x) = \sin\ 2x$. Then

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$$\begin{aligned} f(x+\pi) &= \sin\ 2\,(x+\pi) \\ &= \sin\,(2x+2\pi) \\ &= \sin\ 2x \quad \text{(as we showed above)} \\ &= f(x) \end{aligned}$$

for all x, so the period p of $\sin\ 2x$ is *at most* π, by our definition of period. We have to show that $p>0$ can not be smaller than π. To do this, we will use a *proof by contradiction*. That is, assume that $0<p<\pi$, then show that this leads to some contradiction, and hence can not be true. So suppose $0<p<\pi$. Then $0<2p<2\pi$, and hence

$$\begin{aligned} \sin\ 2x &= f(x) \\ &= f(x+p) \quad (\text{since } p \text{ is the period of } f(x)) \\ &= \sin\ 2(x+p) \\ &= \sin\,(2x+2p) \end{aligned}$$

for all x. Since any number u can be written as $2x$ for some x (i.e $u=2(u/2)$), this means that $\sin\ u=\sin\,(u+2p)$ for all real numbers u, and hence the period of $\sin\ x$ is as most $2p$. This is a contradiction. Why? Because the period of $\sin\ x$ is $2\pi>2p$. Hence, the period p of $\sin\ 2x$ can not be less than π, so the period must equal π.

The above may seem like a lot of work to prove something that was visually obvious from the graph (and intuitively obvious by the "twice as fast" idea). Luckily, we do not need to go through all that work for each function, since a similar argument works when $\sin\ 2x$ is replaced by $\sin\ \omega x$ for any positive real number ω: instead of dividing 2π by 2 to get the period, divide by ω. And the argument works for the other trigonometric functions as well. Thus, we get:

For any number $\omega>0$:

$$\begin{aligned} &\sin\ \omega x \ \text{ has period } \frac{2\pi}{\omega} && \csc\ \omega x \ \text{ has period } \frac{2\pi}{\omega} \\ &\cos\ \omega x \ \text{ has period } \frac{2\pi}{\omega} && \sec\ \omega x \ \text{ has period } \frac{2\pi}{\omega} \\ &\tan\ \omega x \ \text{ has period } \frac{\pi}{\omega} && \cot\ \omega x \ \text{ has period } \frac{\pi}{\omega} \end{aligned}$$

If $\omega<0$, then use $\sin\,(-A)=-\sin\ A$ and $\cos\,(-A)=\cos\ A$ (e.g. $\sin\,(-3x)=-\sin\ 3x$).

Example 5.2.3

The period of $y=\cos\ 3x$ is $\frac{2\pi}{3}$ and the period of $y=\cos\ \frac{1}{2}x$ is 4π. The graphs of both functions are shown in Figure 5.2.2:

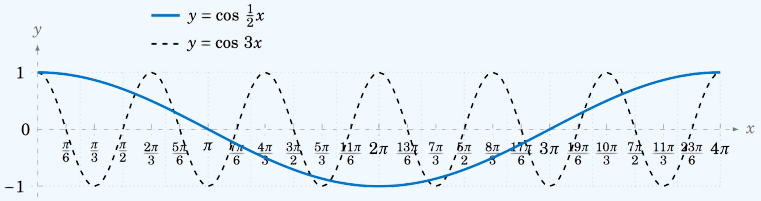

Figure 5.2.2: Graph of $y=\cos 3x$ and $y=\cos\ \frac{1}{2}x$

We know that $-1\le\sin\ x\le 1$ and $-1\le\cos\ x\le 1$ for all x. Thus, for a constant $A\neq 0$,

$$-|A|\ \le\ A\sin\ x\ \le\ |A| \quad \text{and} \quad -|A|\ \le\ A\cos\ x\ \le\ |A|$$

for all x. In this case, we call $|A|$ the **amplitude** of the functions $y=A\sin\ x$ and $y=A\cos\ x$. In general, the amplitude of a periodic curve $f(x)$ is half the difference of the largest and smallest values that $f(x)$ can take:

$$\text{Amplitude of } f(x)\ =\ \frac{(\text{maximum of } f(x))\ -\ (\text{minimum of } f(x))}{2}$$

In other words, the amplitude is the distance from either the top or bottom of the curve to the horizontal line that divides the curve in half, as in Figure 5.2.3.

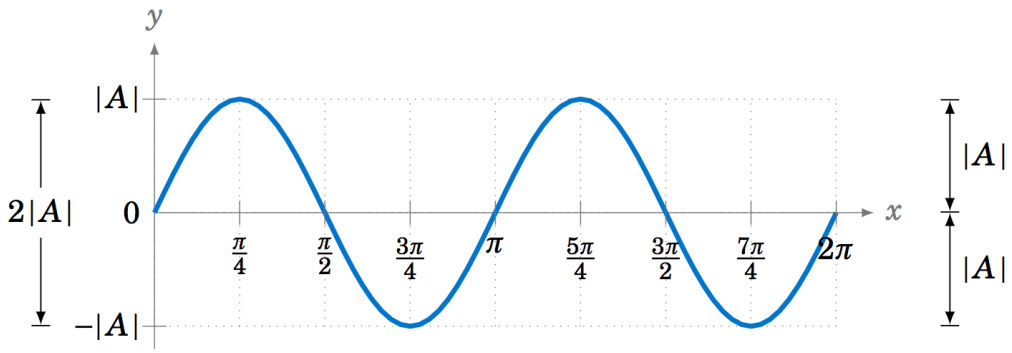

Figure 5.2.3 Amplitude = $\frac{\max-\min}{2} = \frac{|A|-(-|A|)}{2} = |A|$

Not all periodic curves have an amplitude. For example, $\tan\ x$ has neither a maximum nor a minimum, so its amplitude is undefined. Likewise, $\cot\ x$, $\csc\ x$, and $\sec\ x$ do not have an amplitude. Since the amplitude involves vertical distances, it has no effect on the period of a function, and vice versa.

Example 5.2.4

Find the amplitude and period of $y = 3\ \cos\ 2x$.

Solution

The amplitude is $|3| = 3$ and the period is $\frac{2\pi}{2} = \pi$. The graph is shown in Figure 5.2.4:

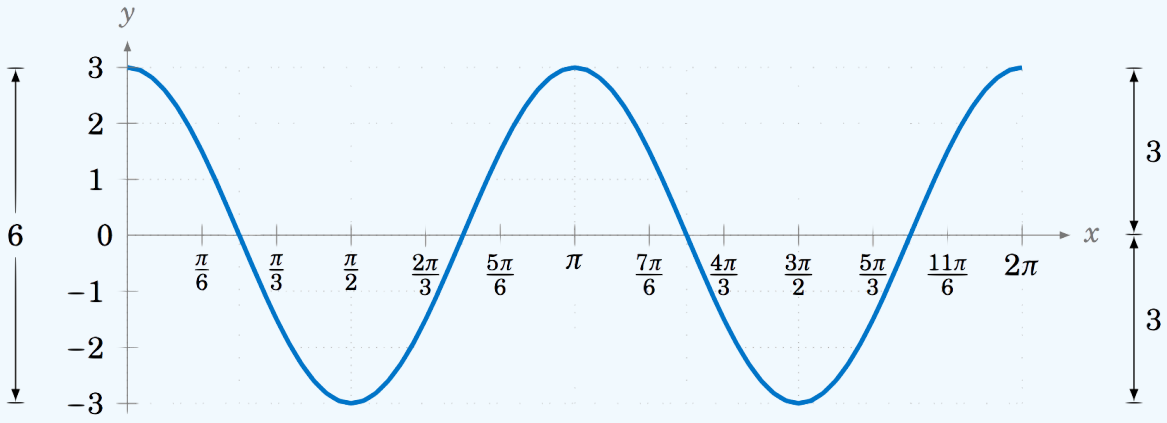

Figure 5.2.4 $y = 3\cos 2x$

Example 5.2.5

Find the amplitude and period of $y = 2 - 3\ \sin\ \frac{2\pi}{3}x$.

Solution

The amplitude of $-3\ \sin\ \frac{2\pi}{3}x$ is $|-3| = 3$. Adding 2 to that function to get the function $y = 2 - 3\ \sin\ \frac{2\pi}{3}x$ does not change the amplitude, even though it does change the maximum and minimum. It just shifts the entire graph upward by 2. So in this case, we have

$$\text{Amplitude} = \frac{\max - \min}{2} = \frac{5 - (-1)}{2} = \frac{6}{2} = 3\,.$$

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The period is $\frac{2\pi}{\frac{2\pi}{3}} = 3$. The graph is shown in Figure 5.2.5:

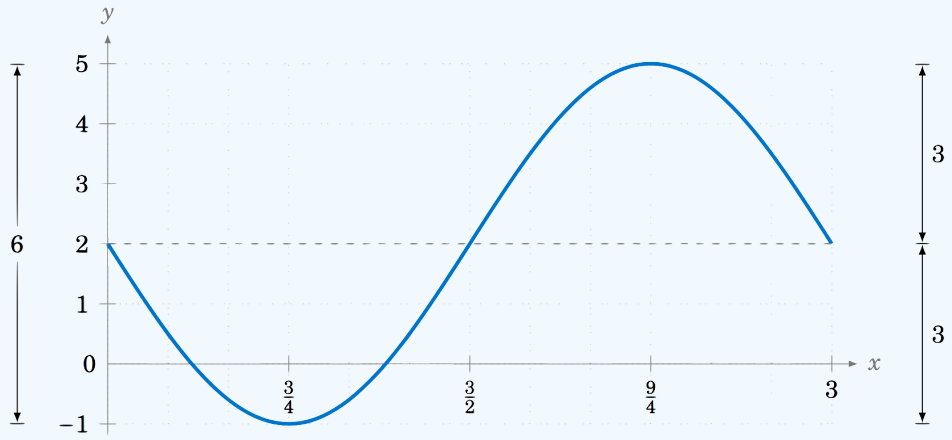


Figure 5.2.5 $y = 2 - 3\sin\frac{2\pi}{3}x$

Example 5.2.6

Find the amplitude and period of $y = 2\ \sin\ (x^2)$.

Solution

This is not a periodic function, since the angle that we are taking the sine of, x^2, is not a *linear* function of x, i.e. is not of the form $ax + b$ for some constants a and b. Recall how we argued that $\sin\ 2x$ was ``twice as fast" as $\sin\ x$, so that its period was π instead of 2π. Can we say that $\sin\ (x^2)$ is some *constant* times as fast as $\sin\ x$? No. In fact, we see that the "speed" of the curve keeps increasing as x gets larger, since x^2 grows at a variable rate, not a constant rate. This can be seen in the graph of $y = 2\ \sin\ (x^2)$, shown in Figure 5.2.6:

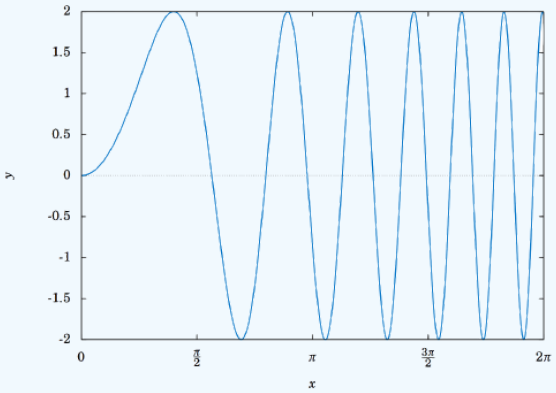


Figure 5.2.6 $y = 2\sin(x^2)$

Notice how the curve "speeds up" as x gets larger, making the "waves" narrower and narrower. Thus, $y = 2\ \sin\ (x^2)$ has no period. Despite this, it appears that the function does have an amplitude, namely 2. To see why, note that since $|\sin\ \theta| \leq 1$ for all θ, we have

$$|2\ \sin\ (x^2)|\ =\ |2|\ \cdot\ |\sin\ (x^2)|\ \leq\ 2\ \cdot\ 1\ =\ 2\ .$$

In the exercises you will be asked to find values of x such that $2\ \sin\ (x^2)$ reaches the maximum value 2 and the minimum value -2. Thus, the amplitude is indeed 2.

Note: This curve is still sinusoidal despite not being periodic, since the general shape is still that of a "sine wave", albeit one with variable *cycles*.

So far in our examples we have been able to determine the amplitudes of sinusoidal curves fairly easily. This will not always be the case.

Example 5.2.7

Find the amplitude and period of $y = 3\sin x + 4\cos x$.

Solution

This is sometimes called a *combination* sinusoidal curve, since it is the sum of two such curves. The period is still simple to determine: since $\sin x$ and $\cos x$ each repeat every 2π radians, then so does the combination $3\sin x + 4\cos x$. Thus, $y = 3\sin x + 4\cos x$ has period 2π. We can see this in the graph, shown in Figure 5.2.7:

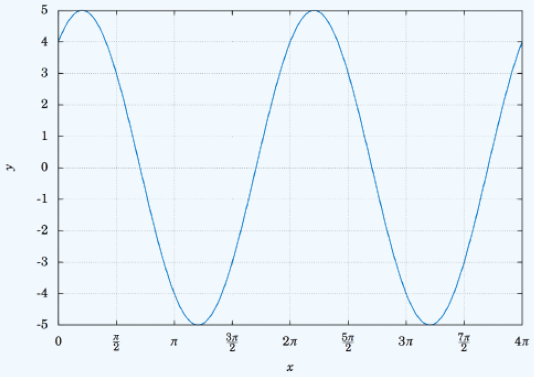


Figure 5.2.7 $y = 3\sin x + 4\cos x$

The graph suggests that the amplitude is 5, which may not be immediately obvious just by looking at how the function is defined. In fact, the definition $y = 3\sin x + 4\cos x$ may tempt you to think that the amplitude is 7, since the largest that $3\sin x$ could be is 3 and the largest that $4\cos x$ could be is 4, so that the largest their sum could be is $3 + 4 = 7$. However, $3\sin x$ can never equal 3 for the same x that makes $4\cos x$ equal to 4 (why?).

There is a useful technique (which we will discuss further in Chapter 6) for showing that the amplitude of $y = 3\sin x + 4\cos x$ is 5. Let θ be the angle shown in the right triangle in Figure 5.2.8. Then $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$. We can use this as follows:

$$\begin{aligned} y &= 3\sin x + 4\cos x \\ &= 5\left(\tfrac{3}{5}\sin x + \tfrac{4}{5}\cos x\right) \\ &= 5\left(\cos\theta\sin x + \sin\theta\cos x\right) \\ &= 5\sin(x+\theta) \quad \text{(by the sine addition formula)} \end{aligned}$$

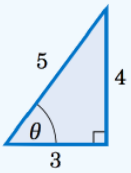


Figure 5.2.8

Thus, $|y| = |5\sin(x+\theta)| = |5| \cdot |\sin(x+\theta)| \le (5)(1) = 5$, so the amplitude of $y = 3\sin x + 4\cos x$ is 5.

In general, a combination of sines and cosines will have a period equal to the *lowest common multiple* of the periods of the sines and cosines being added. In Example 5.9, $\sin x$ and $\cos x$ each have period 2π, so the lowest common multiple (which is always an *integer* multiple) is $1 \cdot 2\pi = 2\pi$.

Example 5.2.8

Find the period of $y = \cos 6x + \sin 4x$.

Solution

The period of $\cos\ 6x$ is $\frac{2\pi}{6} = \frac{\pi}{3}$, and the period of $\sin\ 4x$ is $\frac{2\pi}{4} = \frac{\pi}{2}$. The lowest common multiple of $\frac{\pi}{3}$ and $\frac{\pi}{2}$ is π:

$$\begin{aligned} 1 \cdot \tfrac{\pi}{3} &= \tfrac{\pi}{3} & \qquad 1 \cdot \tfrac{\pi}{2} &= \tfrac{\pi}{2} \\ 2 \cdot \tfrac{\pi}{3} &= \tfrac{2\pi}{3} & \qquad 2 \cdot \tfrac{\pi}{2} &= \pi \\ 3 \cdot \tfrac{\pi}{3} &= \pi \end{aligned}$$

Thus, the period of $y = \cos\ 6x + \sin\ 4x$ is π. We can see this from its graph in Figure 5.2.9:

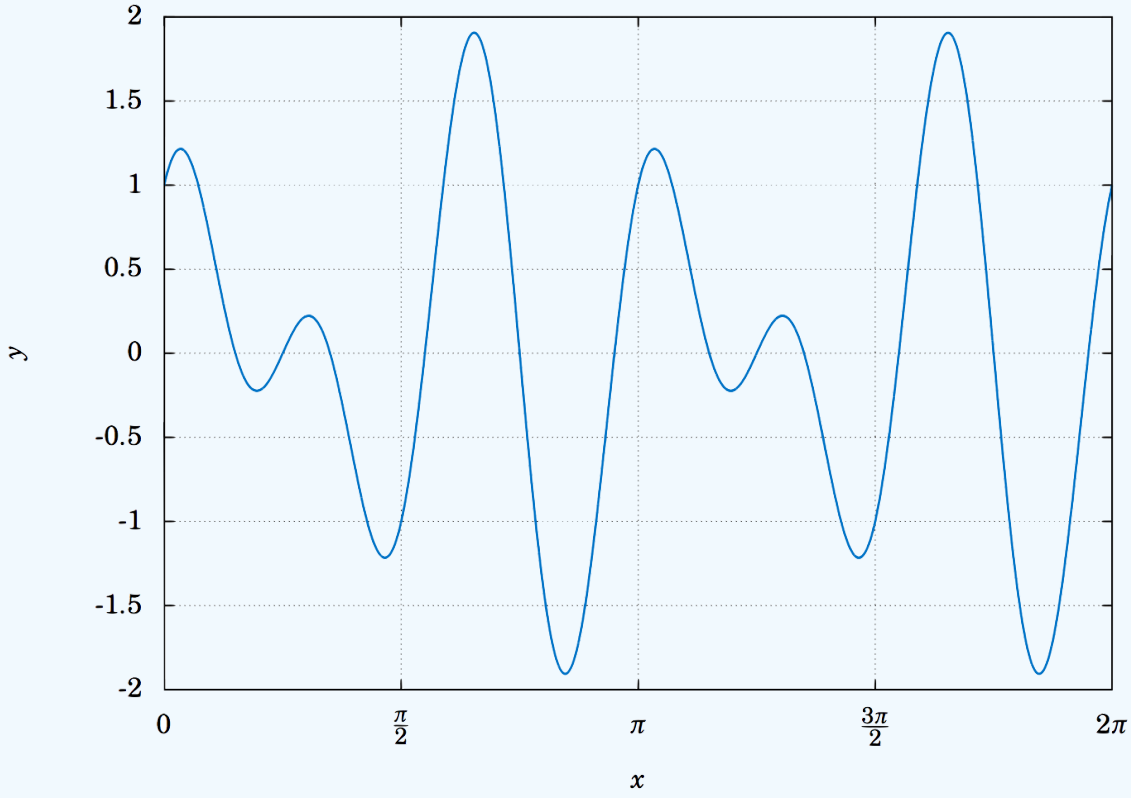


Figure 5.2.9 $y = \cos 6x + \sin 4x$

What about the amplitude? Unfortunately we can not use the technique from Example 5.9, since we are not taking the cosine and sine of the same angle; we are taking the cosine of $6x$ but the sine of $4x$. In this case, it appears from the graph that the maximum is close to 2 and the minimum is close to -2. In Chapter 6, we will describe how to use a numerical computation program to show that the maximum and minimum are ± 1.90596111871578 respectively (accurate to within $\approx 2.2204 \times 10^{-16}$). Hence, the amplitude is 1.90596111871578

Generalizing Example 5.9, an expression of the form $a\ \sin\ \omega x\ +\ b\ \cos\ \omega x$ is equivalent to $\sqrt{a^2+b^2}\ \sin\ (x+\theta)$, where θ is an angle such that $\cos\ \theta = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\ \theta = \frac{b}{\sqrt{a^2+b^2}}$. So $y = a\ \sin\ \omega x\ +\ b\ \cos\ \omega x$ will have amplitude $\sqrt{a^2+b^2}$. Note that this method only works when the angle ωx is the same in both the sine and cosine terms.

We have seen how adding a constant to a function shifts the entire graph vertically. We will now see how to shift the entire graph of a periodic curve horizontally.

Consider a function of the form $y = A\ \sin\ \omega x$, where A and ω are nonzero constants. For simplicity we will assume that $A > 0$ and $\omega > 0$ (in general either one could be negative). Then the amplitude is A and the period is $\frac{2\pi}{\omega}$. The graph is shown in Figure 5.2.10.

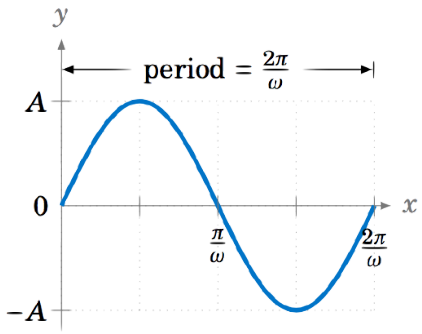


Figure 5.2.10 $y = A \sin \omega x$

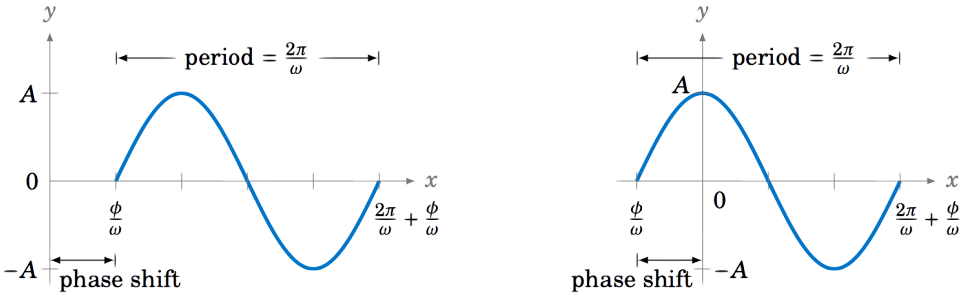
Now consider the function $y = A \sin(\omega x - \phi)$, where ϕ is some constant. The amplitude is still A, and the period is still $\frac{2\pi}{\omega}$, since $\omega x - \phi$ is a linear function of x. Also, we know that the sine function goes through an entire cycle when its angle goes from 0 to 2π. Here, we are taking the sine of the angle $\omega x - \phi$. So as $\omega x - \phi$ goes from 0 to 2π, an entire cycle of the function $y = A \sin(\omega x - \phi)$ will be traced out. That cycle starts when

$$\omega x - \phi = 0 \quad \Rightarrow \quad x = \frac{\phi}{\omega}$$

and ends when

$$\omega x - \phi = 2\pi \quad \Rightarrow \quad x = \frac{2\pi}{\omega} + \frac{\phi}{\omega}.$$

Thus, the graph of $y = A \sin(\omega x - \phi)$ is just the graph of $y = A \sin \omega x$ shifted horizontally by $\frac{\phi}{\omega}$, as in Figure 5.2.11. The graph is shifted to the right when $\phi > 0$, and to the left when $\phi < 0$. The amount $\frac{\phi}{\omega}$ of the shift is called the **phase shift** of the graph.



(a) $\phi > 0$: right shift **(b)** $\phi < 0$: left shift

Figure 5.2.11 Phase shift for $y = A \sin(\omega x - \varphi)$

The phase shift is defined similarly for the other trigonometric functions.

Example 5.2.9

Find the amplitude, period, and phase shift of $y = 3 \cos(2x - \pi)$.

Solution

The amplitude is 3, the period is $\frac{2\pi}{2} = \pi$, and the phase shift is $\frac{\pi}{2}$. The graph is shown in Figure 5.2.12:

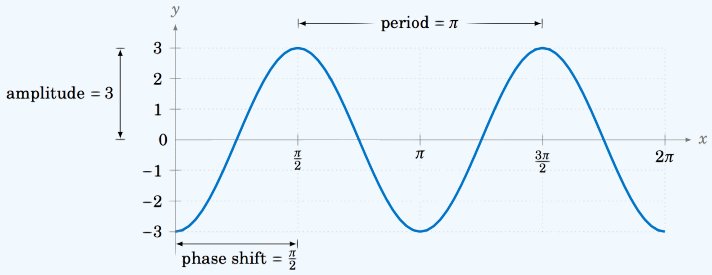


Figure 5.2.12 $y = 3\cos(2x - \pi)$

Notice that the graph is the same as the graph of $y = 3\ \cos\ 2x$ shifted to the right by $\frac{\pi}{2}$, the amount of the phase shift.

Example 5.2.10

Find the amplitude, period, and phase shift of $y = -2\ \sin\ \left(3x + \frac{\pi}{2}\right)$.

Solution

The amplitude is 2, the period is $\frac{2\pi}{3}$, and the phase shift is $\frac{-\frac{\pi}{2}}{3} = -\frac{\pi}{6}$. Notice the negative sign in the phase shift, since $3x + \pi = 3x - (-\pi)$ is in the form $\omega x - \phi$. The graph is shown in Figure 5.2.13:

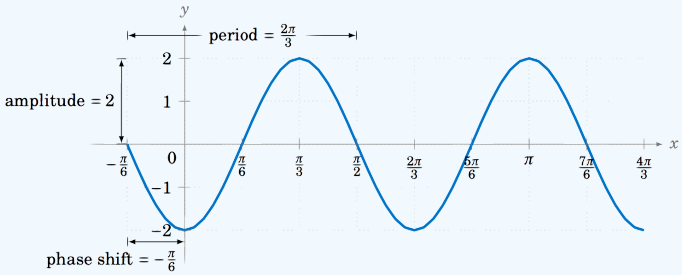


Figure 5.2.13 $y = -2\sin(3x + \pi 2)$

In engineering two periodic functions with the same period are said to be *out of phase* if their phase shifts differ. For example, $\sin\ \left(x - \frac{\pi}{6}\right)$ and $\sin\ x$ would be $\frac{\pi}{6}$ radians (or 30°) out of phase, and $\sin\ x$ would be said to *lag* $\sin\ \left(x - \frac{\pi}{6}\right)$ by $\frac{\pi}{6}$ radians, while $\sin\ \left(x - \frac{\pi}{6}\right)$ *leads* \(\sin\;x \) by $\frac{\pi}{6}$ radians. Periodic functions with the same period and the same phase shift are *in phase*.

The following is a summary of the properties of trigonometric graphs:

For any constants $A \neq 0$, $\omega \neq 0$, and ϕ:

$$y = A\ \sin\ (\omega x - \phi)\ \text{ has amplitude } |A|, \text{ period } \frac{2\pi}{\omega}, \text{ and phase shift } \frac{\phi}{\omega}$$
$$y = A\ \cos\ (\omega x - \phi)\ \text{ has amplitude } |A|, \text{ period } \frac{2\pi}{\omega}, \text{ and phase shift } \frac{\phi}{\omega}$$
$$y = A\ \tan\ (\omega x - \phi)\ \text{ has undefined amplitude, period } \frac{\pi}{\omega}, \text{ and phase shift } \frac{\phi}{\omega}$$
$$y = A\ \csc\ (\omega x - \phi)\ \text{ has undefined amplitude, period } \frac{2\pi}{\omega}, \text{ and phase shift } \frac{\phi}{\omega}$$
$$y = A\ \sec\ (\omega x - \phi)\ \text{ has undefined amplitude, period } \frac{2\pi}{\omega}, \text{ and phase shift } \frac{\phi}{\omega}$$
$$y = A\ \cot\ (\omega x - \phi)\ \text{ has undefined amplitude, period } \frac{\pi}{\omega}, \text{ and phase shift } \frac{\phi}{\omega}$$

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5.3: Inverse Trigonometric Functions

We have briefly mentioned the inverse trigonometric functions before, for example in Section 1.3 when we discussed how to use the $\boxed{\sin^{-1}}$, $\boxed{\cos^{-1}}$, and $\boxed{\tan^{-1}}$ buttons on a calculator to find an angle that has a certain trigonometric function value. We will now define those inverse functions and determine their graphs.

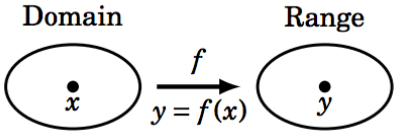

Figure 5.3.1

Recall that a **function** is a rule that assigns a single object y from one set (the **range** to each object x from another set (the **domain**). We can write that rule as $y = f(x)$, where f is the function (see Figure 5.3.1). There is a simple *vertical rule* for determining whether a rule $y = f(x)$ is a function: f is a function if and only if every vertical line intersects the graph of $y = f(x)$ in the xy-coordinate plane at most once (see Figure 5.3.2).

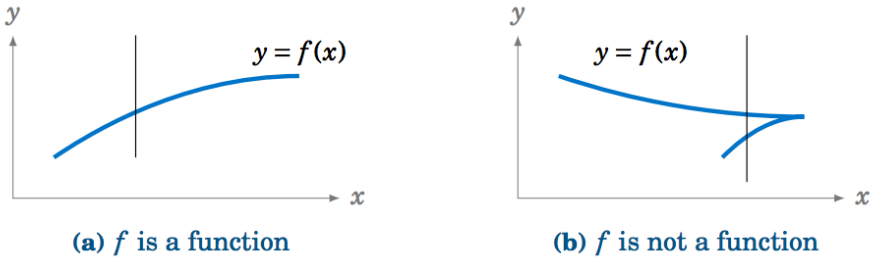

Figure 5.3.2 Vertical rule for functions

Recall that a function f is **one-to-one** (often written as $1-1$) if it assigns distinct values of y to distinct values of x. In other words, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. Equivalently, f is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. There is a simple *horizontal rule* for determining whether a function $y = f(x)$ is one-to-one: f is one-to-one if and only if every horizontal line intersects the graph of $y = f(x)$ in the xy-coordinate plane at most once (see Figure 5.3.3).

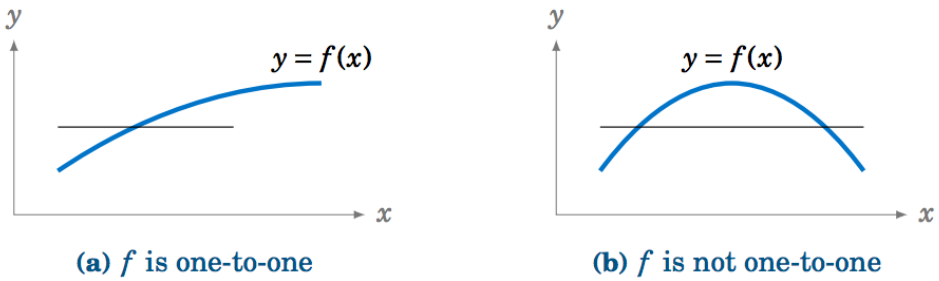

Figure 5.3.3 Horizontal rule for one-to-one functions

If a function f is one-to-one on its domain, then f has an **inverse function**, denoted by f^{-1}, such that $y = f(x)$ if and only if $f^{-1}(y) = x$. The domain of f^{-1} is the range of f.

The basic idea is that f^{-1} "undoes" what f does, and vice versa. In other words,

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for all } x \text{ in the domain of } f, \text{ and} \\ f(f^{-1}(y)) &= y \quad \text{for all } y \text{ in the range of } f. \end{aligned}$$

We know from their graphs that none of the trigonometric functions are one-to-one over their entire domains. However, we can restrict those functions to *subsets* of their domains where they *are* one-to-one. For example, $y = \sin\ x$ is one-to-one over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, as we see in the graph below:

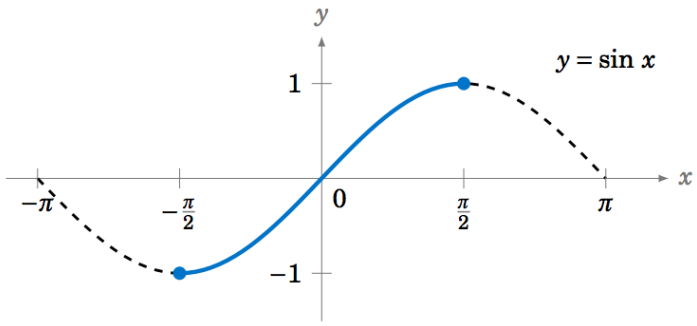

Figure 5.3.4 $y = \sin x$ with x restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

For $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ we have $-1 \le \sin\ x \le 1$, so we can define the **inverse sine** function $y = \sin^{-1} x$ (sometimes called the **arc sine** and denoted by $y = \arcsin\ x$) whose domain is the interval $[-1, 1]$ and whose range is the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In other words:

$$\sin^{-1}(\sin\ y) = y \quad \text{for } -\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2} \tag{5.3.1}$$
$$\sin\ (\sin^{-1} x) = x \quad \text{for } -1 \le x \le 1 \tag{5.3.2}$$

Example 5.13

Find $\sin^{-1}\left(\sin\ \frac{\pi}{4}\right)$.

Solution

Since $-\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}$, we know that $\sin^{-1}\left(\sin\ \frac{\pi}{4}\right) = \boxed{\frac{\pi}{4}}$, by Equation 5.3.1.

Example 5.14

Find $\sin^{-1}\left(\sin\ \frac{5\pi}{4}\right)$.

Solution

Since $\frac{5\pi}{4} > \frac{\pi}{2}$, we can not use Equation 5.3.1. But we know that $\sin\ \frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$. Thus, $\sin^{-1}\left(\sin\ \frac{5\pi}{4}\right) = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is, by definition, the angle y such that $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $\sin\ y = -\frac{1}{\sqrt{2}}$. That angle is $y = -\frac{\pi}{4}$, since

$$\sin\ \left(-\tfrac{\pi}{4}\right) = -\sin\ \left(\tfrac{\pi}{4}\right) = -\tfrac{1}{\sqrt{2}}.$$

Thus, $\sin^{-1}\left(\sin\ \frac{5\pi}{4}\right) = \boxed{-\frac{\pi}{4}}$.

Example 5.14 illustrates an important point: $\sin^{-1} x$ should *always* be a number between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. If you get a number outside that range, then you made a mistake somewhere. This why in Example 1.27 in Section 1.5 we got $\sin^{-1}(-0.682) = -43^\circ$ when using the $\boxed{\sin^{-1}}$ button on a calculator. Instead of an angle between 0° and 360° (i.e. 0 to 2π radians) we got an angle between -90° and 90° (i.e. $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ radians).

In general, the graph of an inverse function f^{-1} is the reflection of the graph of f around the line $y = x$. The graph of $y = \sin^{-1} x$ is shown in Figure 5.3.5. Notice the symmetry about the line $y = x$ with the graph of $y = \sin\ x$.

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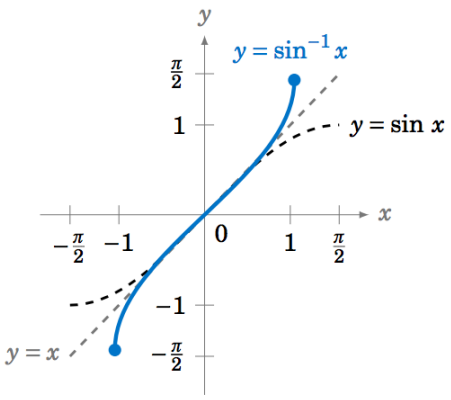

Figure 5.3.5 Graph of $y = \sin^{-1} x$

The **inverse cosine** function $y = \cos^{-1} x$ (sometimes called the **arc cosine** and denoted by $y = \arccos\ x$) can be determined in a similar fashion. The function $y = \cos\ x$ is one-to-one over the interval $[0, \pi]$, as we see in the graph below:

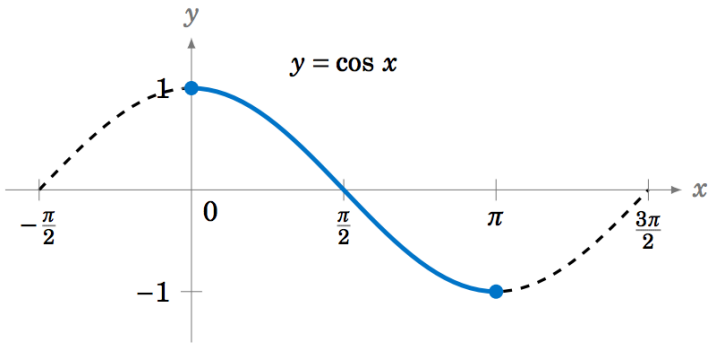

Figure 5.3.6 $y = \cos x$ with x restricted to $[0, \pi]$

Thus, $y = \cos^{-1} x$ is a function whose domain is the interval $[-1, 1]$ and whose range is the interval $[0, \pi]$. In other words:

$$\cos^{-1}(\cos\ y) = y \quad \text{for } 0 \le y \le \pi \tag{5.3.3}$$

$$\cos\ (\cos^{-1} x) = x \quad \text{for } -1 \le x \le 1 \tag{5.3.4}$$

The graph of $y = \cos^{-1} x$ is shown below in Figure 5.3.7. Notice the symmetry about the line $y = x$ with the graph of $y = \cos\ x$.

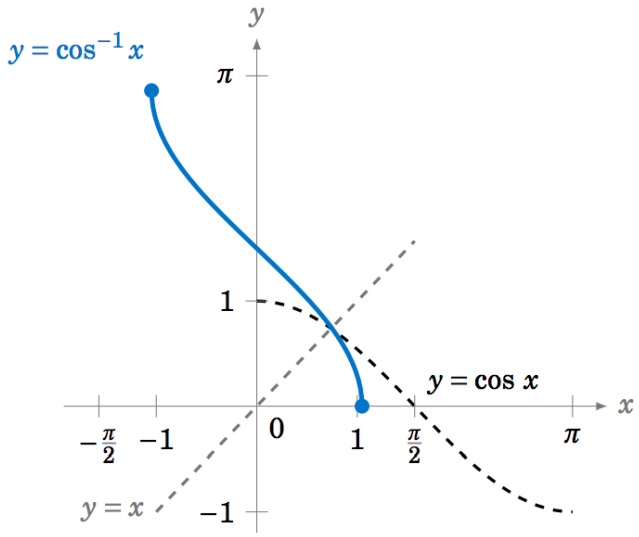

Figure 5.3.7 Graph of $y = \cos^{-1} x$

Example 5.15

Find $\cos^{-1}\left(\cos\ \frac{\pi}{3}\right)$.

Solution

Since $0 \leq \frac{\pi}{3} \leq \pi$, we know that $\cos^{-1}\left(\cos\ \frac{\pi}{3}\right) = \boxed{\frac{\pi}{3}}$, by Equation 5.3.3.

Example 5.16

Find $\cos^{-1}\left(\cos\ \frac{4\pi}{3}\right)$.

Solution

Since $\frac{4\pi}{3} > \pi$, we can not use Equation 5.3.3. But we know that $\cos\ \frac{4\pi}{3} = -\frac{1}{2}$. Thus, $\cos^{-1}\left(\cos\ \frac{4\pi}{3}\right) = \cos^{-1}\left(-\frac{1}{2}\right)$ is, by definition, the angle y such that $0 \leq y \leq \pi$ and $\cos\ y = -\frac{1}{2}$. That angle is $y = \frac{2\pi}{3}$ (i.e. 120°). Thus, $\cos^{-1}\left(\cos\ \frac{4\pi}{3}\right) = \boxed{\frac{2\pi}{3}}$.

Examples 5.14 and 5.16 may be confusing, since they seem to violate the general rule for inverse functions that $f^{-1}(f(x)) = x$ for all x in the domain of f. But that rule only applies when the function f is one-to-one over its *entire* domain. We had to restrict the sine and cosine functions to very small subsets of their entire domains in order for those functions to be one-to-one. That general rule, therefore, only holds for x in those small subsets in the case of the inverse sine and inverse cosine.

The **inverse tangent** function $y = \tan^{-1} x$ (sometimes called the **arc tangent** and denoted by $y = \arctan\ x$) can be determined similarly. The function $y = \tan\ x$ is one-to-one over the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, as we see in Figure 5.3.8:

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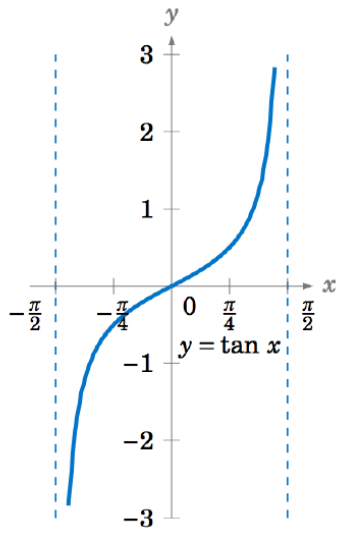

Figure 5.3.8 $y = \tan x$ with x restricted to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

The graph of $y = \tan^{-1} x$ is shown below in Figure 5.3.9. Notice that the vertical asymptotes for $y = \tan\ x$ become horizontal asymptotes for $y = \tan^{-1} x$. Note also the symmetry about the line $y = x$ with the graph of $y = \tan\ x$.

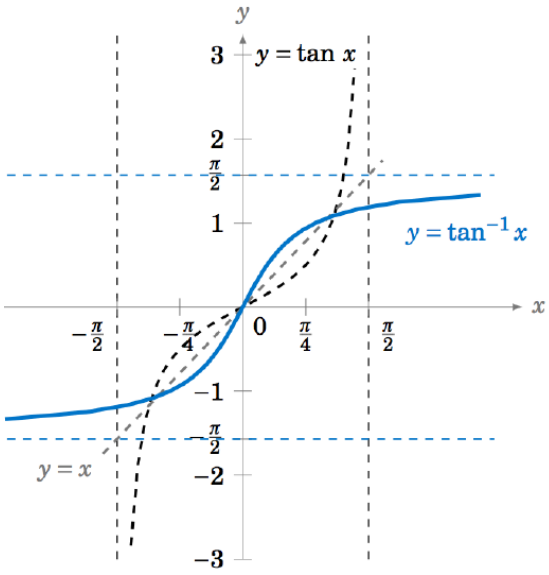

Figure 5.3.9 Graph of $y = \tan^{-1} x$

Thus, $y = \tan^{-1} x$ is a function whose domain is the set of all real numbers and whose range is the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In other words:

$$\tan^{-1}(\tan\ y) = y \quad \text{for } -\tfrac{\pi}{2} < y < \tfrac{\pi}{2} \tag{5.3.5}$$

$$\tan\ (\tan^{-1} x) = x \quad \text{for all real } x \tag{5.3.6}$$

Example 5.17

Find $\tan^{-1}\left(\tan\ \frac{\pi}{4}\right)$.

Solution

Since $-\frac{\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2}$, we know that $\tan^{-1}\left(\tan\ \frac{\pi}{4}\right) = \boxed{\frac{\pi}{4}}$, by Equation 5.3.5.

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Example 5.18

Find $\tan^{-1}(\tan\ \pi)$.

Solution

Since $\pi > \frac{\pi}{2}$, we can not use Equation 5.3.5. But we know that $\tan\ \pi = 0$. Thus, $\tan^{-1}(\tan\ \pi) = \tan^{-1} 0$ is, by definition, the angle y such that $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $\tan\ y = 0$. That angle is $y = 0$. Thus, $\tan^{-1}(\tan\ \pi) = \boxed{0}$.

Example 5.19

Find the exact value of $\cos\ \left(\sin^{-1}\ \left(-\frac{1}{4}\right)\right)$.

Solution

Let $\theta = \sin^{-1}\ \left(-\frac{1}{4}\right)$. We know that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so since $\sin\ \theta = -\frac{1}{4} < 0$, θ must be in QIV. Hence $\cos\ \theta > 0$. Thus,

$$\cos^2\ \theta\ =\ 1\ -\ \sin^2\ \theta\ =\ 1\ -\ \left(-\frac{1}{4}\right)^2\ =\ \frac{15}{16}\quad \Rightarrow\quad \cos\ \theta\ =\ \frac{\sqrt{15}}{4}\ .$$

Note that we took the positive square root above since $\cos\ \theta > 0$. Thus, $\cos\ \left(\sin^{-1}\ \left(-\frac{1}{4}\right)\right) = \boxed{\frac{\sqrt{15}}{4}}$.

Example 5.20

Show that $\tan\ (\sin^{-1} x) = \dfrac{x}{\sqrt{1-x^2}}$ for $-1 < x < 1$.

Solution

When $x = 0$, the Equation holds trivially, since

$$\tan\ (\sin^{-1} 0)\ =\ \tan\ 0\ =\ 0\ =\ \frac{0}{\sqrt{1-0^2}}\ .$$

Now suppose that $0 < x < 1$. Let $\theta = \sin^{-1} x$. Then θ is in QI and $\sin\ \theta = x$. Draw a right triangle with an angle θ such that the opposite leg has length x and the hypotenuse has length 1, as in Figure 5.3.10 (note that this is possible since $0 < x < 1$). Then $\sin\ \theta = \frac{x}{1} = x$. By the Pythagorean Theorem, the adjacent leg has length $\sqrt{1-x^2}$. Thus, $\tan\ \theta = \frac{x}{\sqrt{1-x^2}}$.

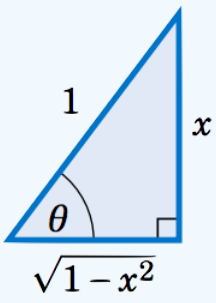

Figure 5.3.10

If $-1 < x < 0$ then $\theta = \sin^{-1} x$ is in QIV. So we can draw the same triangle except that it would be "upside down" and we would again have $\tan\ \theta = \frac{x}{\sqrt{1-x^2}}$, since the tangent and sine have the same sign (negative) in QIV. Thus, $\tan\ (\sin^{-1} x) = \dfrac{x}{\sqrt{1-x^2}}$ for $-1 < x < 1$.

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The inverse functions for cotangent, cosecant, and secant can be determined by looking at their graphs. For example, the function $y = \cot\ x$ is one-to-one in the interval $(0, \pi)$, where it has a range equal to the set of all real numbers. Thus, the **inverse cotangent** $y = \cot^{-1} x$ is a function whose domain is the set of all real numbers and whose range is the interval $(0, \pi)$. In other words:

$$\cot^{-1}(\cot\ y) = y \quad \text{for } 0 < y < \pi \tag{5.3.7}$$
$$\cot\ (\cot^{-1} x) = x \quad \text{for all real } x \tag{5.3.8}$$

The graph of $y = \cot^{-1} x$ is shown below in Figure 5.3.11.

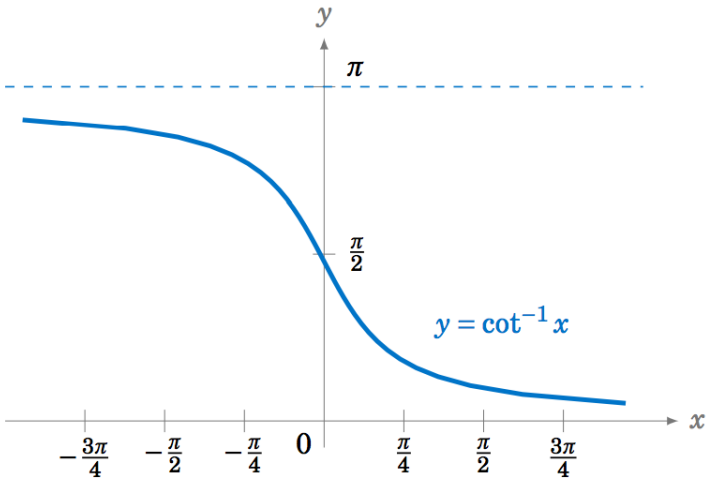


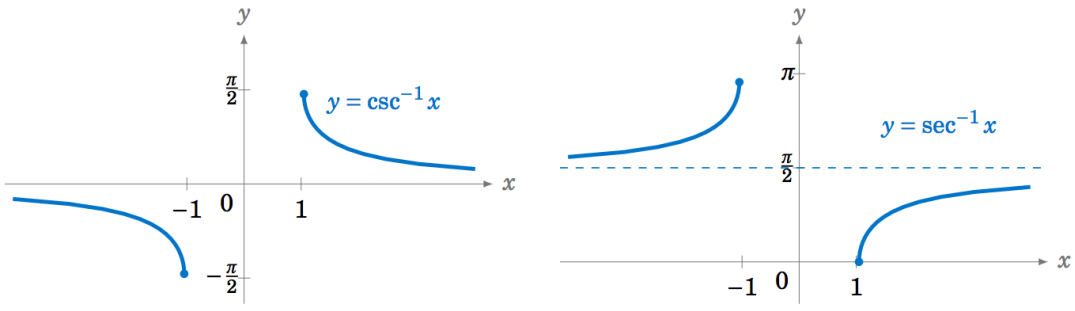
Figure 5.3.11 Graph of $y = \cot^{-1} x$

Similarly, it can be shown that the **inverse cosecant** $y = \csc^{-1} x$ is a function whose domain is $|x| \geq 1$ and whose range is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$. Likewise, the **inverse secant** $y = \sec^{-1} x$ is a function whose domain is $|x| \geq 1$ and whose range is $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$.

$$\csc^{-1}(\csc\ y) = y \quad \text{for } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0 \tag{5.3.9}$$
$$\csc\ (\csc^{-1} x) = x \quad \text{for } |x| \geq 1 \tag{5.3.10}$$

$$\sec^{-1}(\sec\ y) = y \quad \text{for } 0 \leq y \leq \pi, y \neq \frac{\pi}{2} \tag{5.3.11}$$
$$\sec\ (\sec^{-1} x) = x \quad \text{for } |x| \geq 1 \tag{5.3.12}$$

It is also common to call $\cot^{-1} x$, $\csc^{-1} x$, and $\sec^{-1} x$ the **arc cotangent**, **arc cosecant**, and **arc secant**, respectively, of x. The graphs of $y = \csc^{-1} x$ and $y = \sec^{-1} x$ are shown in Figure 5.3.12:



(a) Graph of $y = \csc^{-1} x$ **(b) Graph of $y = \sec^{-1} x$**

Figure 5.3.12

Example 5.21

Prove the identity $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$.

Solution:

Let $\theta = \cot^{-1} x$. Using relations from Section 1.5, we have

$$\tan\left(\tfrac{\pi}{2} - \theta\right) = -\tan\left(\theta - \tfrac{\pi}{2}\right) = \cot\,\theta = \cot\left(\cot^{-1} x\right) = x\,,$$

by Equation 5.3.8. So since $\tan\left(\tan^{-1} x\right) = x$ for all x, this means that $\tan\left(\tan^{-1} x\right) = \tan\left(\frac{\pi}{2} - \theta\right)$. Thus, $\tan\left(\tan^{-1} x\right) = \tan\left(\frac{\pi}{2} - \cot^{-1} x\right)$. Now, we know that $0 < \cot^{-1} x < \pi$, so $-\frac{\pi}{2} < \frac{\pi}{2} - \cot^{-1} x < \frac{\pi}{2}$, i.e. $\frac{\pi}{2} - \cot^{-1} x$ is in the restricted subset on which the tangent function is one-to-one. Hence, $\tan\left(\tan^{-1} x\right) = \tan\left(\frac{\pi}{2} - \cot^{-1} x\right)$ implies that $\tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x$, which proves the identity.

Example 5.22

Is $\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\dfrac{a+b}{1-ab}\right)$ an identity?

Solution

In the tangent addition Equation $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$, let $A = \tan^{-1} a$ and $B = \tan^{-1} b$. Then

$$\begin{aligned}\tan\left(\tan^{-1} a + \tan^{-1} b\right) &= \frac{\tan\left(\tan^{-1} a\right) + \tan\left(\tan^{-1} b\right)}{1 - \tan\left(\tan^{-1} a\right)\tan\left(\tan^{-1} b\right)} \\ &= \frac{a+b}{1-ab} \qquad \text{by Equation 5.3.6, so it seems that we have} \\ \tan^{-1} a + \tan^{-1} b &= \tan^{-1}\left(\frac{a+b}{1-ab}\right)\end{aligned}$$

by definition of the inverse tangent. However, recall that $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$ for all real numbers x. So in particular, we must have $-\frac{\pi}{2} < \tan^{-1}\left(\frac{a+b}{1-ab}\right) < \frac{\pi}{2}$. But it is possible that $\tan^{-1} a + \tan^{-1} b$ is *not* in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. For example,

$$\tan^{-1} 1 + \tan^{-1} 2 = 1.892547 > \tfrac{\pi}{2} \approx 1.570796\,.$$

And we see that $\tan^{-1}\left(\frac{1+2}{1-(1)(2)}\right) = \tan^{-1}(-3) = -1.249045 \neq \tan^{-1} 1 + \tan^{-1} 2$. So the Equation is only true when $-\frac{\pi}{2} < \tan^{-1} a + \tan^{-1} b < \frac{\pi}{2}$.

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5.E: Graphing and Inverse Functions (Exercises)

These are homework exercises to accompany Corral's "Elementary Trigonometry" Textmap. This is a text on elementary trigonometry, designed for students who have completed courses in high-school algebra and geometry. Though designed for college students, it could also be used in high schools. The traditional topics are covered, but a more geometrical approach is taken than usual. Also, some numerical methods (e.g. the secant method for solving trigonometric equations) are discussed.

5.1 Exercise

For Exercises 1-12, draw the graph of the given function for $0 \leq x \leq 2\pi$.

5.1.1 $y = -\cos\ x$

5.1.2 $y = 1 + \sin\ x$

5.1.3 $y = 2 - \cos\ x$

5.1.4 $y = 2 - \sin\ x$

5.1.5 $y = -\tan\ x$

5.1.6 $y = -\cot\ x$

5.1.7 $y = 1 + \sec\ x$

5.1.8 $y = -1 - \csc\ x$

5.1.9 $y = 2\sin\ x$

5.1.10 $y = -3\cos\ x$

5.1.11 $y = -2\tan\ x$

5.1.12 $y = -2\sec\ x$

5.1.13 We can extend the unit circle definition of the sine and cosine functions to all six trigonometric functions. Let P be a point in QI on the unit circle, so that the line segment $\overline{OP}$ in Figure 5.1.10 has length 1 and makes an acute angle θ with the positive x-axis. Identify each of the six trigonometric functions of θ with exactly one of the line segments in Figure 5.1.10, keeping in mind that the radius of the circle is 1. To get you started, we have $\sin\ \theta = MP$ (why?).

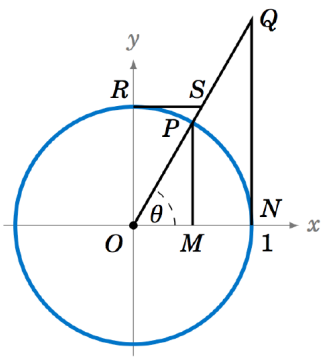

Figure 5.1.10

5.1.14 For Exercise 13, how would you draw the line segments in Figure 5.1.10 if θ was in QII? Recall that some of the trigonometric functions are negative in QII, so you will have to come up with a convention for how to treat some of the line segment lengths as negative.

5.1.15 For any point (x, y) on the unit circle and any angle α, show that the point $R_\alpha(x, y)$ defined by $R_\alpha(x, y) = (x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha)$ is also on the unit circle. What is the geometric interpretation of $R_\alpha(x, y)$? Also, show that $R_{-\alpha}(R_\alpha(x, y)) = (x, y)$ and $R_\beta(R_\alpha(x, y)) = R_{\alpha+\beta}(x, y)$.

5.2 Exercise

For Exercises 1-12, find the amplitude, period, and phase shift of the given function. Then graph one cycle of the function, either by hand or by using Gnuplot (see Appendix B).

5.2.1 $y = 3 \cos \pi x$

5.2.2 $y = \sin(2\pi x - \pi)$

5.2.3 $y = -\sin(5x + 3)$

5.2.4 $y = 1 + 8 \cos(6x - \pi)$

5.2.5 $y = 2 + \cos(5x + \pi)$

5.2.6 $y = 1 - \sin(3\pi - 2x)$

5.2.7 $y = 1 - \cos(3\pi - 2x)$

5.2.8 $y = 2 \tan(x - 1)$

5.2.9 $y = 1 - \tan(3\pi - 2x)$

5.2.10 $y = \sec(2x + 1)$

5.2.11 $y = 2 \csc(2x - 1)$

5.2.12 $y = 2 + 4 \cot(1 - x)$

5.2.13 For the function $y = 2 \sin(x^2)$ in Example 5.8, for which values of x does the function reach its maximum value 2, and for which values of x does it reach its minimum value -2?

5.2.14 For the function $y = 3 \sin x + 4 \cos x$ in Example 5.9, for which values of x does the function reach its maximum value 5, and for which values of x does it reach its minimum value -5? You can restrict your answers to be between 0 and 2π.

5.2.15 Graph the function $y = \sin^2 x$ from $x = 0$ to $x = 2\pi$, either by hand or by using Gnuplot. What are the amplitude and period of this function?

5.2.16 The current $i(t)$ in an AC electrical circuit at time $t \geq 0$ is given by $i(t) = I_m \sin \omega t$, and the voltage $v(t)$ is given by $v(t) = V_m \sin \omega t$, where $V_m > I_m > 0$ and $\omega > 0$ are constants. Sketch one cycle of both $i(t)$ and $v(t)$ *together on the same graph* (i.e. on the same set of axes). Are the current and voltage in phase or out of phase?

5.2.17 Repeat Exercise 16 with $i(t)$ the same as before but with $v(t) = V_m \sin\left(\omega t + \frac{\pi}{4}\right)$.

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5.2.18 Repeat Exercise 16 with $i(t) = -I_m \cos\left(\omega t - \frac{\pi}{3}\right)$ and $v(t) = V_m \sin\left(\omega t - \frac{5\pi}{6}\right)$.

For Exercises 19-21, find the amplitude and period of the given function. Then graph one cycle of the function, either by hand or by using Gnuplot.

5.2.19 $y = 3 \sin \pi x - 5 \cos \pi x$

5.2.20 $y = -5 \sin 3x + 12 \cos 3x$

5.2.21 $y = 2 \cos x + 2 \sin x$

5.2.22 Find the amplitude of the function $y = 2 \sin(x^2) + \cos(x^2)$.

For Exercises 23-25, find the period of the given function. Graph one cycle using Gnuplot.

5.2.23 $y = \sin 3x - \cos 5x$

5.2.24 $y = \sin \frac{x}{3} + 2 \cos \frac{3x}{4}$

5.2.25 $y = 2 \sin \pi x + 3 \cos \frac{\pi}{3} x$

5.2.26 Let $y = 0.5 \sin x \sin 12x$. Its graph for x from 0 to 4π is shown in Figure 5.2.14:

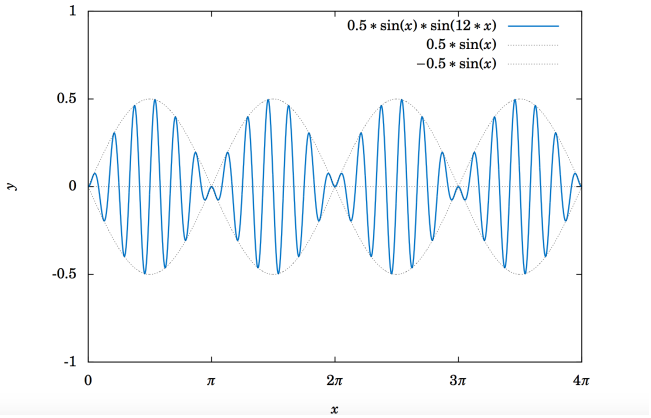

Figure 5.2.14 Modulated wave $y = 0.5 \sin x \sin 12x$

You can think of this function as $\sin 12x$ with a sinusoidally varying "amplitude"of $0.5 \sin x$. What is the period of this function? From the graph it looks like the amplitude may be 0.5. Without finding the exact amplitude, explain why the amplitude is in fact *less* than 0.5. The function above is known as a *modulated wave*, and the functions $\pm 0.5 \sin x$ form an *amplitude envelope* for the wave (i.e. they enclose the wave). Use an identity from Section 3.4 to write this function as a sum of sinusoidal curves.

5.2.27 Use Gnuplot to graph the function $y = x^2 \sin 10x$ from $x = -2\pi$ to $x = 2\pi$. What functions form its amplitude envelope? (Note: Use `set samples 500` in Gnuplot.)

5.2.28 Use Gnuplot to graph the function $y = \frac{1}{x^2} \sin 80x$ from $x = 0.2$ to $x = \pi$. What functions form its amplitude envelope? (Note: Use `set samples 500` in Gnuplot.)

5.2.29 Does the function $y = \sin \pi x + \cos x$ have a period? Explain your answer.

5.2.30 Use Gnuplot to graph the function $y = \frac{\sin\ x}{x}$ from $x = -4\pi$ to $x = 4\pi$. What happens at $x = 0$?

5.3 Exercise

For Exercises 1-25, find the exact value of the given expression in radians.

5.3.1 $\tan^{-1} 1$

5.3.2 $\tan^{-1} (-1)$

5.3.3 $\tan^{-1} 0$

5.3.4 $\cos^{-1} 1$

5.3.5 $\cos^{-1} (-1)$

5.3.6 $\cos^{-1} 0$

5.3.7 $\sin^{-1} 1$

5.3.8 $\sin^{-1} (-1)$

5.3.9 $\sin^{-1} 0$

5.3.10 $\sin^{-1} \left(\sin\ \frac{\pi}{3}\right)$

5.3.11 $\sin^{-1} \left(\sin\ \frac{4\pi}{3}\right)$

5.3.12 $\sin^{-1} \left(\sin\ \left(-\frac{5\pi}{6}\right)\right)$

5.3.13 $\cos^{-1} \left(\cos\ \frac{\pi}{7}\right)$

5.3.14 $\cos^{-1} \left(\cos\ \left(-\frac{\pi}{10}\right)\right)$

5.3.15 $\cos^{-1} \left(\cos\ \frac{6\pi}{5}\right)$

5.3.16 $\tan^{-1} \left(\tan\ \frac{4\pi}{3}\right)$

5.3.17 $\tan^{-1} \left(\tan\ \left(-\frac{5\pi}{6}\right)\right)$

5.3.18 $\cot^{-1} \left(\cot\ \frac{4\pi}{3}\right)$

5.3.19 $\csc^{-1} \left(\csc\ \left(-\frac{\pi}{9}\right)\right)$

5.3.20 $\sec^{-1} \left(\sec\ \frac{6\pi}{5}\right)$

5.3.21 $\cos\ \left(\sin^{-1}\ \left(\frac{5}{13}\right)\right)$

5.3.22 $\cos\ \left(\sin^{-1}\ \left(-\frac{4}{5}\right)\right)$

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5.3.23 $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{4}{5}$

5.3.24 $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{5}{13}$

5.3.25 $\tan^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{5}$

For Exercises 26-33, prove the given identity.

5.3.26 $\cos\left(\sin^{-1}x\right)=\sqrt{1-x^2}$

5.3.27 $\sin\left(\cos^{-1}x\right)=\sqrt{1-x^2}$

5.3.28 $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$

5.3.29 $\sec^{-1}x+\csc^{-1}x=\frac{\pi}{2}$

5.3.30 $\sin^{-1}(-x)=-\sin^{-1}x$

5.3.31 $\cos^{-1}(-x)+\cos^{-1}x=\pi$

5.3.32 $\cot^{-1}x=\tan^{-1}\frac{1}{x}$ for $x>0$

5.3.33 $\tan^{-1}x+\tan^{-1}\frac{1}{x}=\frac{\pi}{2}$ for $x>0$

5.3.34 In Example 5.22 we showed that the formula $\tan^{-1}a+\tan^{-1}b=\tan^{-1}\left(\frac{a+b}{1-ab}\right)$ does not always hold. Does the formula $\tan\left(\tan^{-1}a+\tan^{-1}b\right)=\frac{a+b}{1-ab}$, which was part of that example, always hold? Explain your answer.

5.3.35 Show that $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{4}{7}$.

5.3.36 Show that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\tan^{-1}\frac{1}{2}$.

5.3.37 Figure 5.3.13 shows three equal squares lined up against each other. For the angles α, β, and γ in the picture, show that $\alpha=\beta+\gamma$. (*Hint: Consider the tangents of the angles.*)

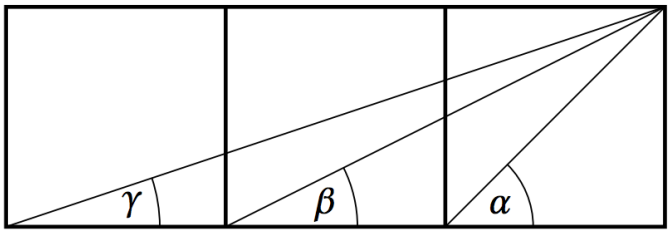

Figure 5.3.13 Exercise 37

5.3.38 Sketch the graph of $y=\sin^{-1}2x$.

5.3.39 Write a computer program to solve a triangle in the case where you are given three sides. Your program should read in the three sides as input parameters and print the three angles in degrees as output if a solution exists. Note that since most computer languages use radians for their inverse trigonometric functions, you will likely have to do the conversion from radians to degrees yourself in the program.

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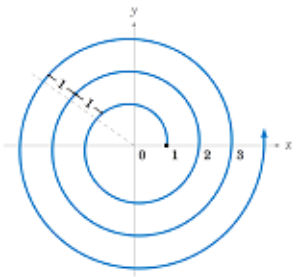
CHAPTER OVERVIEW

6: ADDITIONAL TOPICS



6.1: SOLVING TRIGONOMETRIC EQUATIONS

In Chapter 1 we were concerned only with finding a single solution (say, between 0° and 90°). In this section we will be concerned with finding the most general solution to such trigonometric equations.

6.2: NUMERICAL METHODS IN TRIGONOMETRY

We were able to solve the trigonometric equations in the previous section fairly easily, which in general is not the case. Instead, we have to resort to numerical methods, which provide ways of getting successively better approximations to the actual solution(s) to within any desired degree of accuracy.

6.3: COMPLEX NUMBERS

There is no real number x such that $x^2 = -1$. However, it turns out to be useful to invent such a number, called the imaginary unit and denoted by the letter i.

6.4: POLAR COORDINATES

Suppose that from the point (1,0) in the xy-coordinate plane we draw a spiral around the origin, such that the distance between any two points separated by 360° along the spiral is always 1, as in Figure 6.4.1. We can not express this spiral as $y = f(x)$ for some function f in Cartesian coordinates, since its graph violates the vertical rule. However, this spiral would be simple to describe using the polar coordinate system.

6.E: ADDITIONAL TOPICS (EXERCISES)

These are homework exercises to accompany Corral's "Elementary Trigonometry" Textmap. This is a text on elementary trigonometry, designed for students who have completed courses in high-school algebra and geometry. Though designed for college students, it could also be used in high schools. The traditional topics are covered, but a more geometrical approach is taken than usual. Also, some numerical methods (e.g. the secant method for solving trigonometric equations) are discussed.

6.1: Solving Trigonometric Equations

An equation involving trigonometric functions is called a *trigonometric equation.* For example, an equation like

$$\tan\ A\ =\ 0.75\ ,$$

which we encountered in Chapter 1, is a trigonometric equation. In Chapter 1 we were concerned only with finding a single solution (say, between 0° and 90°). In this section we will be concerned with finding the most general solution to such equations.

To see what that means, take the above equation $\tan\ A = 0.75$. Using the $\boxed{\tan^{-1}}$ calculator button in degree mode, we get $A = 36.87^\circ$. However, we know that the tangent function has period π rad $= 180^\circ$, i.e. it repeats every 180°. Thus, there are many other possible answers for the value of A, namely $36.87^\circ + 180^\circ$, $36.87^\circ - 180^\circ$, $36.87^\circ + 360^\circ$, $36.87^\circ - 360^\circ$, $36.87^\circ + 540^\circ$, etc. We can write this in a more compact form:

$$A\ =\ 36.87^\circ\ +\ 180^\circ k \quad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

This is the most general solution to the equation. Often the part that says "for $k = 0, \pm 1, \pm 2, \ldots$" is omitted since it is usually understood that k varies over all integers. The general solution in radians would be:

$$A\ =\ 0.6435\ +\ \pi k \quad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

Example 6.1

Solve the equation $2\ \sin\ \theta\ +\ 1\ =\ 0$.

Solution:

Isolating $\sin\ \theta$ gives $\sin\ \theta\ =\ -\frac{1}{2}$. Using the $\boxed{\sin^{-1}}$ calculator button in degree mode gives us $\theta = -30^\circ$, which is in QIV. Recall that the reflection of this angle around the y-axis into QIII also has the same sine. That is, $\sin\ 210^\circ = -\frac{1}{2}$. Thus, since the sine function has period 2π rad $= 360^\circ$, and since -30° does not differ from 210° by an integer multiple of 360°, the general solution is:

$$\boxed{\theta\ =\ -30^\circ\ +\ 360^\circ k \quad \text{and} \quad 210^\circ\ +\ 360^\circ k} \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

In radians, the solution is:

$$\boxed{\theta\ =\ -\frac{\pi}{6}\ +\ 2\pi k \quad \text{and} \quad \frac{7\pi}{6} + 2\pi k} \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

For the rest of this section we will write our solutions in radians.

Example 6.2

Solve the equation $2\cos^2\ \theta\ -\ 1\ =\ 0$.

Solution:

Isolating $\cos^2\ \theta$ gives us

$$\cos^2\ \theta\ =\ \frac{1}{2} \quad \Rightarrow \quad \cos\ \theta\ =\ \pm\ \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta\ =\ \frac{\pi}{4}\ ,\ \frac{3\pi}{4}\ ,\ \frac{5\pi}{4}\ ,\ \frac{7\pi}{4}\ ,$$

and since the period of cosine is 2π, we would add $2\pi k$ to each of those angles to get the general solution. But notice that

the above angles differ by multiples of $\frac{\pi}{2}$. So since every multiple of 2π is also a multiple of $\frac{\pi}{2}$, we can combine those four separate answers into one:

$$\boxed{\theta = \frac{\pi}{4} + \frac{\pi}{2}k} \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

Example 6.3

Solve the equation $2 \sec \theta = 1$.

Solution:

Isolating $\sec \theta$ gives us

$$\sec \theta = \frac{1}{2} \quad \Rightarrow \quad \cos \theta = \frac{1}{\sec \theta} = 2\,,$$

which is impossible. Thus, there is $\boxed{\text{no solution}}$.

Example 6.4

Solve the equation $\cos \theta = \tan \theta$.

Solution:

The idea here is to use identities to put everything in terms of a single trigonometric function:

$$\begin{aligned} \cos \theta &= \tan \theta \\ \cos \theta &= \frac{\sin \theta}{\cos \theta} \\ \cos^2 \theta &= \sin \theta \\ 1 - \sin^2 \theta &= \sin \theta \\ 0 &= \sin^2 \theta + \sin \theta - 1 \end{aligned}$$

The last equation looks more complicated than the original equation, but notice that it is actually a *quadratic* equation: making the substitution $x = \sin \theta$, we have

$$x^2 + x - 1 = 0 \quad \Rightarrow \quad x = \frac{-1 \pm \sqrt{1-(4)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2} = -1.618\,,\ 0.618$$

by the quadratic formula from elementary algebra. But $-1.618 < -1$, so it is impossible that $\sin\theta = x = -1.618$. Thus, we must have $\sin \theta = x = 0.618$. Hence, there are two possible solutions: $\theta = 0.666$ rad in QI and its reflection $\pi - \theta = 2.475$ rad around the y-axis in QII. Adding multiples of 2π to these gives us the general solution:

$$\boxed{\theta = 0.666 + 2\pi k \quad \text{and} \quad 2.475 + 2\pi k} \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

Example 6.5

Solve the equation $\sin \theta = \tan \theta$.

Solution:

Trying the same method as in the previous example, we get

$$\begin{aligned} \sin\theta &= \tan\theta \\ \sin\theta &= \frac{\sin\theta}{\cos\theta} \\ \sin\theta\cos\theta &= \sin\theta \\ \sin\theta\cos\theta - \sin\theta &= 0 \\ \sin\theta(\cos\theta - 1) &= 0 \\ &\Rightarrow \quad \sin\theta = 0 \quad \text{or} \quad \cos\theta = 1 \\ &\Rightarrow \quad \theta = 0, \pi \quad \text{or} \quad \theta = 0 \\ &\Rightarrow \quad \theta = 0, \pi, \end{aligned}$$

plus multiples of 2π. So since the above angles are multiples of π, and every multiple of 2π is a multiple of π, we can combine the two answers into one for the general solution:

$$\boxed{\theta = \pi k} \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

Example 6.6

Solve the equation $\cos 3\theta = \frac{1}{2}$.

Solution:

The idea here is to solve for 3θ first, using the most general solution, and then divide that solution by 3. So since $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$, there are two possible solutions for 3θ: $3\theta = \frac{\pi}{3}$ in QI and its reflection $-3\theta = -\frac{\pi}{3}$ around the x-axis in QIV. Adding multiples of 2π to these gives us:

$$3\theta = \pm\frac{\pi}{3} + 2\pi k \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

So dividing everything by 3 we get the general solution for θ:

$$\boxed{\theta = \pm\frac{\pi}{9} + \frac{2\pi}{3}k} \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

Example 6.7

Solve the equation $\sin 2\theta = \sin\theta$.

Solution:

Here we use the double-angle formula for sine:

$$\begin{aligned} \sin 2\theta &= \sin\theta \\ 2\sin\theta\cos\theta &= \sin\theta \\ \sin\theta(2\cos\theta - 1) &= 0 \\ &\Rightarrow \quad \sin\theta = 0 \quad \text{or} \quad \cos\theta = \frac{1}{2} \\ &\Rightarrow \quad \theta = 0, \pi \quad \text{or} \quad \theta = \pm\frac{\pi}{3} \\ &\Rightarrow \quad \boxed{\theta = \pi k \quad \text{and} \quad \pm\frac{\pi}{3} + 2\pi k} \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

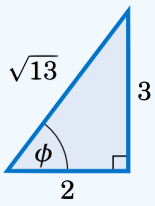


Figure 6.1.1

Solution:

We will use the technique which we discussed in Chapter 5 for finding the amplitude of a combination of sine and cosine functions. Take the coefficients 2 and 3 of $\sin\,\theta$ and $-\cos\,\theta$, respectively, in the above equation and make them the legs of a right triangle, as in Figure 6.1.1. Let ϕ be the angle shown in the right triangle. The leg with length $3>0$ means that the angle ϕ is above the x-axis, and the leg with length $2>0$ means that ϕ is to the right of the y-axis. Hence, ϕ must be in QI. The hypotenuse has length $\sqrt{13}$ by the Pythagorean Theorem, and hence $\cos\,\phi=\frac{2}{\sqrt{13}}$ and $\sin\,\theta=\frac{3}{\sqrt{13}}$. We can use this to transform the equation to solve as follows:

$$\begin{aligned} 2\sin\,\theta-3\cos\,\theta &= 1 \\ \sqrt{13}\left(\tfrac{2}{\sqrt{13}}\sin\,\theta-\tfrac{3}{\sqrt{13}}\cos\,\theta\right) &= 1 \\ \sqrt{13}\,(\cos\,\phi\,\sin\,\theta-\sin\,\phi\,\cos\,\theta) &= 1 \\ \sqrt{13}\,\sin\,(\theta-\phi) &= 1 \quad \text{(by the sine subtraction formula)} \\ \sin\,(\theta-\phi) &= \tfrac{1}{\sqrt{13}} \\ &\Rightarrow \quad \theta-\phi=0.281 \quad \text{or} \quad \theta-\phi=\pi-0.281=2.861 \\ &\Rightarrow \quad \theta=\phi+0.281 \quad \text{or} \quad \theta=\phi+2.861 \end{aligned}$$

Now, since $\cos\,\phi=\frac{2}{\sqrt{13}}$ and ϕ is in QI, the most general solution for ϕ is $\phi=0.983+2\pi k$ for $k=0,\pm1,\pm2,\ldots$. So since we needed to add multiples of 2π to the solutions 0.281 and 2.861 anyway, the most general solution for θ is:

$$\begin{aligned} \theta &= 0.983+0.281+2\pi k \quad \text{and} \quad 0.983+2.861+2\pi k \\ &\Rightarrow \quad \boxed{\theta=1.264+2\pi k \quad \text{and} \quad 3.844+2\pi k} \quad \text{for } k=0,\pm1,\pm2,\ldots \end{aligned}$$

Note: In Example 6.8 if the equation had been $2\sin\,\theta+3\cos\,\theta=1$ then we still would have used a right triangle with legs of lengths 2 and 3, but we would have used the sine addition formula instead of the subtraction formula.

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6.2: Numerical Methods in Trigonometry

We were able to solve the trigonometric equations in the previous section fairly easily, which in general is not the case. For example, consider the equation

$$\cos\ x\ =\ x\ . \tag{6.2.1}$$

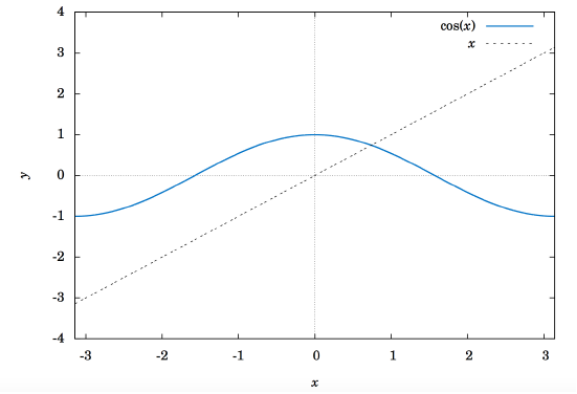

Figure 6.2.1 $y = \cos x$ and $y = x$

Unfortunately there is no trigonometric identity or simple method which will help us here. Instead, we have to resort to *numerical methods*, which provide ways of getting successively better approximations to the actual solution(s) to within any desired degree of accuracy. There is a large field of mathematics devoted to this subject called *numerical analysis*. Many of the methods require calculus, but luckily there is a method which we can use that requires just basic algebra. It is called the *secant method*, and it finds roots of a given function $f(x)$, i.e. values of x such that $f(x) = 0$. A derivation of the secant method is beyond the scope of this book, but we can state the algorithm it uses to solve $f(x) = 0$:

1. Pick initial points x_0 and x_1 such that $x_0 < x_1$ and $f(x_0)\, f(x_1) < 0$ (i.e. the solution is somewhere between x_0 and x_1).
2. For $n \geq 2$, define the number x_n by

$$x_n\ =\ x_{n-1}\ -\ \frac{(x_{n-1}\ -\ x_{n-2})\, f(x_{n-1})}{f(x_{n-1})\ -\ f(x_{n-2})} \tag{6.2.2}$$

   as long as $|x_{n-1}\ -\ x_{n-2}| > \epsilon_{error}$, where $\epsilon_{error} > 0$ is the maximum amount of error desired (usually a very small number).
3. The numbers x_0, x_1, x_2, x_3, ... will approach the solution x as we go through more iterations, getting as close as desired.

We will now show how to use this algorithm to solve the equation $\cos\ x = x$. The solution to that equation is the root of the function $f(x) = \cos\ x - x$. And we saw that the solution is somewhere in the interval $[0, 1]$. So pick $x_0 = 0$ and $x_1 = 1$. Then $f(0) = 1$ and $f(1) = -0.4597$, so that $f(x_0)\, f(x_1) < 0$ (we are using radians, of course). Then by definition,

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$$
\begin{aligned}
x_2 &= x_1 - \frac{(x_1 - x_0)\, f(x_1)}{f(x_1) - f(x_0)} \\
&= 1 - \frac{(1 - 0)\, f(1)}{f(1) - f(0)} \\
&= 1 - \frac{(1 - 0)\,(-0.4597)}{-0.4597 - 1} \\
&= 0.6851\,, \\
x_3 &= x_2 - \frac{(x_2 - x_1)\, f(x_2)}{f(x_2) - f(x_1)} \\
&= 0.6851 - \frac{(0.6851 - 1)\, f(0.6851)}{f(0.6851) - f(1)} \\
&= 0.6851 - \frac{(0.6851 - 1)\,(0.0893)}{0.0893 - (-0.4597)} \\
&= 0.7363\,,
\end{aligned}
$$

and so on. Using a calculator is not very efficient and will lead to rounding errors. A better way to implement the algorithm is with a computer. Listing 6.1 below shows the code (secant.java) for solving $\cos\ x = x$ with the secant method, using the Java programming language:

Listing 6.1 Program listing for secant.java

```
1  import java.math.*;
2  public class secant {
3   public static void main (String[] args) {
4     double x0 =  Double.parseDouble(args[0]);
5     double x1 =  Double.parseDouble(args[1]);
6     double x = 0;
7     double error = 1.0E-50;
8     for (int i=2; i <= 10; i++) {
9        if (Double.compare(Math.abs(x0 - x1),error) > 0) {
10          x = x1 - (x1 - x0)*f(x1)/(f(x1) - f(x0));
11          x0 = x1;
12          x1 = x;
13          System.out.println("x" + i + " = " + x);
14       } else {
15          break;
16       }
17    }
18    MathContext mc = new MathContext(50);
19    BigDecimal answer = new BigDecimal(x,mc);
20    System.out.println("x = " + answer);
21  }
22 //Define the function f(x)
23  public static double f (double x) {
24    return Math.cos(x) - x;
25  }
26 }
```

Lines 4-5 read in x_0 and x_1 as input parameters to the program.
Line 6 initializes the variable that will eventually hold the solution.
Line 7 sets the maximum error ϵ_{error} to be 1.0×10^{-50}. That is, our final answer will be within that (tiny!) amount of the real solution.
Line 8 starts a loop of 9 iterations of the algorithm, i.e. it will create the successive approximations $x_2, x_3, \ldots, x_{10}$ to the real solution, though in Line 9 we check to see if the two previous approximations differ by less than the maximum error. If they do, we stop (since this means we have an acceptable solution), otherwise we continue.
Line 10 is the main step in the algorithm, creating x_n from x_{n-1} and x_{n-2}.
Lines 11-12 set the new values of x_{n-2} and x_{n-1}, respectively.
Lines 18-20 set the number of decimal places to show in the final answer to 50 (the default is 16) and then print the answer.
Lines 23-24 give the definition of the function $f(x) = \cos\ x - x$.

Below is the result of compiling and running the program using $x_0 = 0$ and $x_1 = 1$:

```
javac secant.java
java secant 0 1
x2 = 0.6850733573260451
x3 = 0.736298997613654
x4 = 0.7391193619116293
x5 = 0.7390851121274639
x6 = 0.7390851332150012
x7 = 0.7390851332151607
x8 = 0.7390851332151607
x = 0.739085133215160672293109200836624950170516967773 44
```

Notice that the program only got up to x_8, not x_{10}. The reason is that the difference between x_8 and x_7 was small enough (less than $\epsilon_{error} = 1.0 \times 10^{-50}$) to stop at x_8 and call that our solution. The last line shows that solution to 50 decimal places.

Since $x = 0.73908513321516...$ is the solution of $\cos x = x$, you would get $\cos(\cos x) = \cos x = x$, so $\cos(\cos(\cos x)) = \cos x = x$, and so on. This number x is called an *attractive fixed point* of the function $\cos x$. No matter where you start, you end up getting ``drawn" to it. Figure 6.2.2 shows what happens when starting at $x = 0$: taking the cosine of 0 takes you to 1, and then successive cosines (indicated by the intersections of the vertical lines with the cosine curve) eventually "spiral" in a rectangular fashion to the fixed point (i.e. the solution), which is the intersection of $y = \cos x$ and $y = x$.

Recall in Example 5.10 in Section 5.2 that we claimed that the maximum and minimum of the function $y = \cos 6x + \sin 4x$ were ± 1.90596111871578 respectively. We can show this by using the open-source program Octave. Octave uses a *successive quadratic programming* method to find the minimum of a function $f(x)$. Finding the maximum of $f(x)$ is the same as finding the minimum of $-f(x)$ then multiplying by -1 (why?). Below we show the commands to run at the Octave command prompt (`octave:n>`) to find the minimum of $f(x) = \cos 6x + \sin 4x$. The command `sqp(3,'f')` says to use $x = 3$ as a first approximation of the number x where $f(x)$ is a minimum.

```
octave:1> format long
octave:2> function y = f(x)
> y = cos(6*x) + sin(4*x)
> endfunction
octave:3> sqp(3,'f')
y = -1.90596111871578
ans = 2.65792064609274
```

The output says that the minimum occurs when $x = 2.65792064609274$ and that the minimum is -1.90596111871578 To find the maximum of $f(x)$, we find the minimum of $-f(x)$ and then take its negative. The command `sqp(2,'f')` says to use $x = 2$ as a first approximation of the number x where $f(x)$ is a maximum.

```
octave:4> function y = f(x)
> y = -cos(6*x) - sin(4*x)
> endfunction
octave:5> sqp(2,'f')
y = -1.90596111871578
ans = 2.05446832062993
```

The output says that the maximum occurs when $x = 2.05446832062993$ and that the maximum is $-(-1.90596111871578) = 1.90596111871578$

Recall from Section 2.4 that Heron's formula is adequate for "typical" triangles, but will often have a problem when used in a calculator with, say, a triangle with two sides whose sum is barely larger than the third side. However, you can get around this problem by using computer software capable of handling numbers with a high degree of precision. Most modern computer programming languages have this capability. For example, in the Python programming language (chosen here for simplicity) the `decimal` module can be used to set any level of precision. Below we show how to get accuracy up to 50 decimal places using Heron's formula for the triangle in Example 2.16 from Section 2.4, by using the python interactive command shell:

```
>>> from decimal import *
>>> getcontext().prec = 50
>>> a = Decimal("1000000")
>>> b = Decimal("999999.9999979")
>>> c = Decimal("0.0000029")
>>> s = (a+b+c)/2
>>> K = s*(s-a)*(s-b)*(s-c)
>>> print Decimal(K).sqrt()
0.99999999999894999999998948749999988961874999982 9
```

(Note: The triple arrow $>>>$ is just a command prompt, not part of the code.) Notice in this case that we do get the correct answer; the high level of precision eliminates the
rounding errors shown by many calculators when using Heron's formula.

Another software option is Sage, a powerful and free open-source mathematics package based on Python. It can be run on your own computer, but it can also be run through a web interface: go to http://sagenb.org to create a free account, then once you register and sign in, click the **New Worksheet** link to start entering commands. For example, to find the solution to $\cos\ x = x$ in the interval $[0, 1]$, enter these commands in the worksheet textfield:

```
x = var('x')
find_root(cos(x) == x, 0,1)
```

Click the **evaluate** link to display the answer: 0.7390851332151559

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6.3: Complex Numbers

There is no real number x such that $x^2 = -1$. However, it turns out to be useful to invent such a number, called the **imaginary unit** and denoted by the letter i. Thus, $i^2 = -1$, and hence $i = \sqrt{-1}$. If a and b are real numbers, then a number of the form $a+bi$ is called a **complex number**, and if $b \neq 0$ then it is called an **imaginary number** (and **pure imaginary** if $a = 0$ and $b \neq 0$). The real number a is called the **real part** of the complex number $a+bi$, and bi is called its **imaginary part**.

What does it mean to add a to bi in the definition $a+bi$ of a complex number, i.e. adding a real number and an imaginary number? You can think of it as a way of *extending* the set of real numbers. If $b = 0$ then $a+bi = a+0i = a$ (since $0i$ is defined as 0), so that every real number is a complex number. The imaginary part bi in $a+bi$ can be thought of as a way of taking the *one-dimensional* set of all real numbers and extending it to a *two-dimensional* set: there is a natural correspondence between a complex number $a+bi$ and a *point* (a, b) in the (two-dimensional) xy-coordinate plane.

Before exploring that correspondence further, we will first state some fundamental properties of and operations on complex numbers:

fundamental properties of and operations on complex numbers

Let $a+bi$ and $c+di$ be complex numbers. Then:

1. $a+bi = c+di$ if and only if $a = c$ and $b = d$ (i.e. the real parts are equal and the imaginary parts are equal)
2. $(a+bi) + (c+di) = (a+c) + (b+d)i$ (i.e. add the real parts together and add the imaginary parts together)
3. $(a+bi) - (c+di) = (a-c) + (b-d)i$
4. $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$
5. $(a+bi)(a-bi) = a^2 + b^2$
6. $\dfrac{a+bi}{c+di} = \dfrac{(ac+bd) + (bc-ad)i}{c^2+d^2}$

The first three items above are just definitions of equality, addition, and subtraction of complex numbers. The last three items can be derived by treating the multiplication and division of complex numbers as you would normally treat factors of real numbers:

$$\begin{aligned}(a+bi)(c+di) &= a(c+di) + bi(c+di) \\ &= ac + adi + bci + bdi^2 = ac + adi + bci + bd(-1) \\ &= (ac-bd) + (ad+bc)i\end{aligned}$$

The fifth item is a special case of the multiplication formula:

$$\begin{aligned}(a+bi)(a-bi) &= ((a)(a)-(b)(-b)) + ((a)(-b)+(b)(a))i \\ &= (a^2+b^2) + (-ab+ba)i = (a^2+b^2) + 0i \\ &= a^2 + b^2\end{aligned}$$

The sixth item comes from using the previous items:

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} \\ &= \frac{(ac-b(-d)) + (a(-d)+bc)i}{c^2+d^2} \\ &= \frac{(ac+bd) + (bc-ad)i}{c^2+d^2}\end{aligned}$$

The **conjugate** $\overline{a+bi}$ of a complex number $a+bi$ is defined as $\overline{a+bi} = a-bi$. Notice that $(a+bi) + \overline{(a+bi)} = 2a$ is a real number, $(a+bi) - \overline{(a+bi)} = 2bi$ is an imaginary number if $b \neq 0$, and $(a+bi)\overline{(a+bi)} = a^2+b^2$ is a real

number. So for a complex number $z = a + bi$, $z\bar{z} = a^2 + b^2$ and thus we can define the **modulus** of z to be $\sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$, which we denote by $|z|$.

Example 6.9

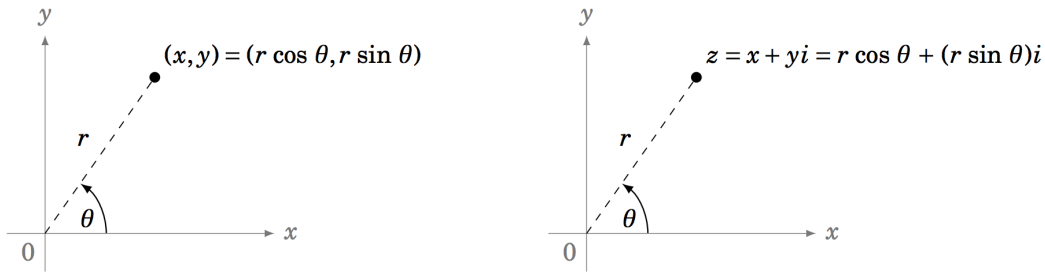
Let $z_1 = -2 + 3i$ and $z_2 = 3 + 4i$. Find $z_1 + z_2$, $z_1 - z_2$, $z_1\, z_2$, z_1/z_2, $|z_1|$, and $|z_2|$.

Solution

Using our rules and definitions, we have:

$$\begin{aligned}
z_1 + z_2 &= (-2+3i) + (3+4i) \\
&= 1+7i \\
z_1 - z_2 &= (-2+3i) - (3+4i) \\
&= -5-i \\
z_1\, z_2 &= (-2+3i)(3+4i) \\
&= ((-2)(3)-(3)(4)) + ((-2)(4)+(3)(3))i \\
&= -18+i \\
\frac{z_1}{z_2} &= \frac{-2+3i}{3+4i} \\
&= \frac{(-2)(3)+(3)(4) + ((3)(3)-(-2)(4))i}{3^2+4^2} \\
&= \frac{6}{25} + \frac{17}{25}\,i \\
|z_1| &= \sqrt{(-2)^2+3^2} \\
&= \sqrt{13} \\
|z_2| &= \sqrt{3^2+4^2} \\
&= 5
\end{aligned}$$

We know that any point (x, y) in the xy-coordinate plane that is a distance $r > 0$ from the origin has coordinates $x = r\cos\theta$ and $y = r\sin\theta$, where θ is the angle in standard position as in Figure 6.3.1(a).



(a) Point (x, y) **(b)** Complex number $z = x + yi$

Figure 6.3.1

Let $z = x + yi$ be a complex number. We can represent z as a point in the **complex plane**, where the horizontal x-axis represents the real part of z, and the vertical y-axis represents the pure imaginary part of z, as in Figure 6.3.1(b). The distance r from z to the origin is, by the Pythagorean Theorem, $r = \sqrt{x^2 + y^2}$, which is just the modulus of z. And we see from Figure 6.3.1(b) that $x = r\cos\theta$ and $y = r\sin\theta$, where θ is the angle formed by the positive x-axis and the line segment from the origin to z. We call this angle θ the **argument** of z. Thus, we get the **trigonometric form** (sometimes called the *polar form*) of the complex number z:

For any complex number $z = x + yi$, we can write

$$z = r(\cos\theta + i\sin\theta)\text{ , where} \tag{6.3.1}$$
$$r = |z| = \sqrt{x^2+y^2}\text{ and}$$
$$\theta = \text{the argument of } z\,.$$

The representation $z = r(\cos\theta + i\sin\theta)$ is often abbreviated as:

$$z = r\,\text{cis}\,\theta \tag{6.3.2}$$

In the special case $z = 0 = 0+0i$, the argument θ is undefined since $r = |z| = 0$. Also, note that the argument θ can be replaced by $\theta + 360^\circ k$ or $\theta + \pi k$, depending on whether you are using degrees or radians, respectively, for $k = 0, \pm 1, \pm 2, \ldots$. Note also that for $z = x+yi$ with $r = |z|$, θ must satisfy

$$\tan\theta = \tfrac{y}{x}\ ,\ \cos\theta = \tfrac{x}{r}\ ,\ \sin\theta = \tfrac{y}{r}\,.$$

Example 6.10

Represent the complex number $-2-i$ in trigonometric form.

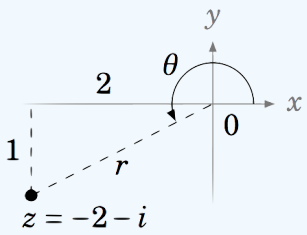


Figure 6.3.2

Solution:

Let $z = -2-i = x+yi$, so that $x = -2$ and $y = -1$. Then θ is in QIII, as we see in Figure 6.3.2. So since $\tan\theta = \frac{y}{x} = \frac{-1}{-2} = \frac{1}{2}$, we have $\theta = 206.6^\circ$. Also,

$$r = \sqrt{x^2+y^2} = \sqrt{(-2)^2+(-1)^2} = \sqrt{5}\,.$$

Thus, $\boxed{-2-i = \sqrt{5}\,(\cos\ 206.6^\circ + i\sin\ 206.6^\circ)}$, or $\sqrt{5}$ cis 206.6°.

For complex numbers in trigonometric form, we have the following Equations for multiplication and division:

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be complex numbers. Then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))\text{ , and} \tag{6.3.3}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) \quad \text{if } z_2 \neq 0. \tag{6.3.4}$$

The proofs of these Equations are straightforward:

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right] \\ &= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \end{aligned}$$

by the addition Equations for sine and cosine. And

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1\left(\cos\,\theta_1\,+\,i\,\sin\,\theta_1\right)}{r_2\left(\cos\,\theta_2\,+\,i\,\sin\,\theta_2\right)}\\ &= \frac{r_1}{r_2}\cdot\frac{\cos\,\theta_1\,+\,i\,\sin\,\theta_1}{\cos\,\theta_2\,+\,i\,\sin\,\theta_2}\cdot\frac{\cos\,\theta_2\,-\,i\,\sin\,\theta_2}{\cos\,\theta_2\,-\,i\,\sin\,\theta_2}\\ &= \frac{r_1}{r_2}\cdot\frac{\left(\cos\,\theta_1\,\cos\,\theta_2\,+\,\sin\,\theta_1\,\sin\,\theta_2\right)\,+\,i\left(\sin\,\theta_1\,\cos\,\theta_2\,-\,\cos\,\theta_1\,\sin\,\theta_2\right)}{\cos^2\,\theta_2\,+\,\sin^2\,\theta_2}\\ &= \frac{r_1}{r_2}\left(\cos\left(\theta_1-\theta_2\right)\,+\,i\,\sin\left(\theta_1-\theta_2\right)\right)\end{aligned}$$

by the subtraction Equations for sine and cosine, and since $\cos^2\,\theta_2\,+\,\sin^2\,\theta_2=1$. **QED**

Note that Equations 6.3.3 and 6.3.4 say that when multiplying complex numbers the moduli are multiplied and the arguments are added, while when dividing complex numbers the moduli are divided and the arguments are subtracted. This makes working with complex numbers in trigonometric form fairly simple.

Example 6.11

Let $z_1=6\left(\cos\,70^\circ\,+\,i\,\sin\,70^\circ\right)$ and $z_1=2\left(\cos\,31^\circ\,+\,i\,\sin\,31^\circ\right)$. Find $z_1\,z_2$ and $\frac{z_1}{z_2}$.

Solution

By Equations 6.3.3 and 6.3.4 we have

$$\begin{aligned}z_1\,z_2 &= (6)\,(2)\left(\cos\left(70^\circ+31^\circ\right)\,+\,i\,\sin\left(70^\circ+31^\circ\right)\right) \quad\Rightarrow\quad \boxed{z_1\,z_2\,=\,12\left(\cos\,101^\circ\,+\,i\,\sin\,101^\circ\right)}\text{, and}\\ \frac{z_1}{z_2} &= \frac{6}{2}\left(\cos\left(70^\circ-31^\circ\right)\,+\,i\,\sin\left(70^\circ-31^\circ\right)\right) \quad\Rightarrow\quad \boxed{\frac{z_1}{z_2}\,=\,3\left(\cos\,39^\circ\,+\,i\,\sin\,39^\circ\right)}\,.\end{aligned}$$

For the special case when $z_1=z_2=z=r\left(\cos\,\theta\,+\,i\,\sin\,\theta\right)$ in Equation 6.3.3, we have

$$\begin{aligned}\left[r\left(\cos\,\theta\,+\,i\,\sin\,\theta\right)\right]^2 &= r\cdot r\left(\cos\left(\theta+\theta\right)\,+\,i\,\sin\left(\theta+\theta\right)\right)\\ &= r^2\left(\cos\,2\theta\,+\,i\,\sin\,2\theta\right),\end{aligned}$$

and so

$$\begin{aligned}\left[r\left(\cos\,\theta\,+\,i\,\sin\,\theta\right)\right]^3 &= \left[r\left(\cos\,\theta\,+\,i\,\sin\,\theta\right)\right]^2\cdot r\left(\cos\,\theta\,+\,i\,\sin\,\theta\right)\\ &= r^2\left(\cos\,2\theta\,+\,i\,\sin\,2\theta\right)\cdot r\left(\cos\,\theta\,+\,i\,\sin\,\theta\right)\\ &= r^3\left(\cos\left(2\theta+\theta\right)\,+\,i\,\sin\left(2\theta+\theta\right)\right)\\ &= r^3\left(\cos\,3\theta\,+\,i\,\sin\,3\theta\right),\end{aligned}$$

and continuing like this (i.e. *by mathematical induction*), we get:

Theorem 6.1 De Moivre's Theorem

For any integer $n\geq 1$,

$$\left[r\left(\cos\,\theta\,+\,i\,\sin\,\theta\right)\right]^n\,=\,r^n\left(\cos\,n\theta\,+\,i\,\sin\,n\theta\right). \tag{6.3.5}$$

We define $z^0=1$ and $z^{-n}=1/z^n$ for all integers $n\geq 1$. So by De Moivre's Theorem and Equation 6.3.3, for any $z=r\left(\cos\,\theta\,+\,i\,\sin\,\theta\right)$ and integer $n\geq 1$ we get

$$\begin{aligned}z^{-n} &= \frac{1}{z^n}\\ &= \frac{1\left(\cos\,0^\circ\,+\,i\,\sin\,0^\circ\right)}{r^n\left(\cos\,n\theta\,+\,i\,\sin\,n\theta\right)}\\ &= \frac{1}{r^n}\left(\cos\left(0^\circ-n\theta\right)\,+\,i\,\sin\left(0^\circ-n\theta\right)\right)\\ &= r^{-n}\left(\cos\left(-n\theta\right)\,+\,i\,\sin\left(-n\theta\right)\right),\end{aligned}$$

and so De Moivre's Theorem in fact holds for *all* integers.

Example 6.12

Find $(1+i)^{10}$.

Solution

Since $1+i=\sqrt{2}\,(\cos\ 45^\circ\ +\ i\ \sin\ 45^\circ)$ (why?), by De Moivre's Theorem we have

$$(1+i)^{10}\ =\ (\sqrt{2})^{10}\,(\cos\ 450^\circ\ +\ i\ \sin\ 450^\circ)\ =\ 2^{10/2}\,(0\ +\ i\,(1))\ =\ 2^5\cdot i\ =\ \boxed{32i}.$$

We can use De Moivre's Theorem to find the n^{th} *roots* of a complex number. That is, given any complex number z and positive integer n, find all complex numbers w such that $w^n=z$. Let $z=r\,(\cos\ \theta\ +\ i\ \sin\ \theta)$. Since the cosine and sine functions repeat every 360°, we know that $z=r\,(\cos\ (\theta+360^\circ k)\ +\ i\ \sin\ (\theta+360^\circ k))$ for $k=0,\pm1,\pm2,\ldots$. Now let $w=r_0\,(\cos\ \theta_0\ +\ i\ \sin\ \theta_0)$ be an n^{th} root of z. Then

$$\begin{aligned} w^n\ =\ z \quad &\Rightarrow \quad [r_0\,(\cos\ \theta_0\ +\ i\ \sin\ \theta_0)]^n\ =\ r\,(\cos\ (\theta+360^\circ k)\ +\ i\ \sin\ (\theta+360^\circ k)) \\ &\Rightarrow \quad r_0^n\,(\cos\ n\theta_0\ +\ i\ \sin\ n\theta_0)\ =\ r\,(\cos\ (\theta+360^\circ k)\ +\ i\ \sin\ (\theta+360^\circ k)) \\ &\Rightarrow \quad r_0^n\ =\ r \quad \text{and} \quad n\theta_0\ =\ \theta+360^\circ k \\ &\Rightarrow \quad r_0\ =\ r^{1/n} \quad \text{and} \quad \theta_0\ =\ \frac{\theta+360^\circ k}{n}\,. \end{aligned}$$

Since the cosine and sine of $\frac{\theta+360^\circ k}{n}$ will repeat for $k\geq n$, we get the following Equation for the n^{th} roots of z:

For any nonzero complex number $z=r\,(\cos\ \theta\ +\ i\ \sin\ \theta)$ and positive integer n, the n distinct n^{th} roots of z are

$$r^{1/n}\left[\cos\left(\frac{\theta+360^\circ k}{n}\right)\ +\ i\ \sin\left(\frac{\theta+360^\circ k}{n}\right)\right] \tag{6.3.6}$$

for $k=0,\,1,\,2,\,\ldots,\,n-1$.

Note: An n^{th} root of z is usually written as $z^{1/n}$ or $\sqrt[n]{z}$. The number $r^{1/n}$ in the above Equation is the usual real n^{th} root of the real number $r=|z|$.

Example 6.13

Find the three cube roots of i.

Solution:

Since $i=1\,(\cos\ 90^\circ\ +\ i\ \sin\ 90^\circ)$, the three cube roots of i are:

$$\begin{aligned} \sqrt[3]{1}\left[\cos\left(\frac{90^\circ+360^\circ(0)}{3}\right)\ +\ i\ \sin\left(\frac{90^\circ+360^\circ(0)}{3}\right)\right]\ &=\ \cos\ 30^\circ\ +\ i\ \sin\ 30^\circ\ =\ \boxed{\frac{\sqrt{3}}{2}\ +\ \frac{1}{2}\,i}, \\ \sqrt[3]{1}\left[\cos\left(\frac{90^\circ+360^\circ(1)}{3}\right)\ +\ i\ \sin\left(\frac{90^\circ+360^\circ(1)}{3}\right)\right]\ &=\ \cos\ 150^\circ\ +\ i\ \sin\ 150^\circ\ =\ \boxed{-\frac{\sqrt{3}}{2}\ +\ \frac{1}{2}\,i}, \\ \sqrt[3]{1}\left[\cos\left(\frac{90^\circ+360^\circ(2)}{3}\right)\ +\ i\ \sin\left(\frac{90^\circ+360^\circ(2)}{3}\right)\right]\ &=\ \cos\ 270^\circ\ +\ i\ \sin\ 270^\circ\ =\ \boxed{-i} \end{aligned}$$

Notice from Example 6.13 that the three cube roots of i are equally spaced points along the unit circle $|z|=1$ in the complex plane, as shown in Figure 6.3.3. We see that consecutive cube roots are 120° apart. In general, the n n^{th} roots of a complex number z will be equally spaced points along the circle of radius $|z|^{1/n}$ in the complex plane, with consecutive roots separated by $\frac{360^\circ}{n}$.

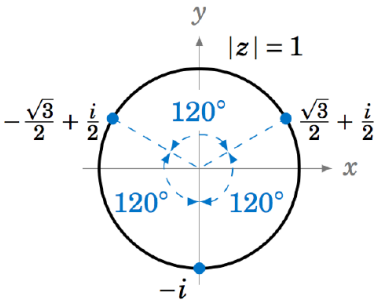


Figure 6.3.3

In higher mathematics the *Fundamental Theorem of Algebra* states that every polynomial of degree n with complex coefficients has n complex roots (some of which may repeat). In particular, every real number a has n n^{th} roots (being the roots of $z^n - a$). For example, the square roots of 1 are ± 1, and the square roots of -1 are $\pm i$.

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6.4: Polar Coordinates

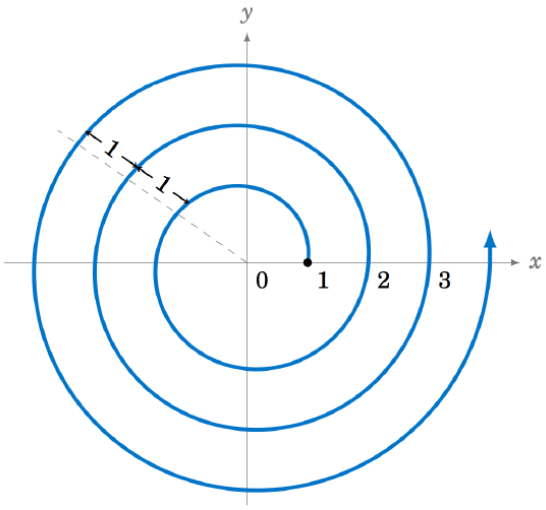

Figure 6.4.2. We call the pair (r,θ) the **polar coordinates** of P, and the positive x-axis is called the **polar axis** of this coordinate system. Note that $(r,\theta)=(r,\theta+360^\circ k)$ for $k=0,\pm1,\pm2,\ldots$, so (unlike for Cartesian coordinates) the polar coordinates of a point are not unique.

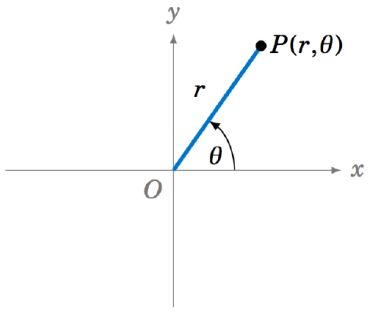

FIGURE 6.4.2 Polar coordinates (r,θ)

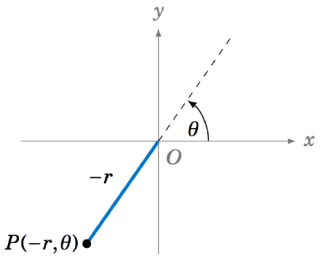

FIGURE 6.4.3 Negative r : $(-r,\theta)$

In polar coordinates we adopt the convention that r can be negative, by defining $(-r,\theta)=(r,\theta+180^\circ)$ for any angle θ. That is, the ray $\overrightarrow{OP}$ is drawn in the opposite direction from the angle θ, as in Figure 6.4.3. When $r=0$, the point $(r,\theta)=(0,\theta)$ is the origin O, regardless of the value of θ.

You may be familiar with graphing paper, for plotting points or functions given in Cartesian coordinates (sometimes also called *rectangular coordinates*). Such paper consists of a rectangular grid. Similar graphing paper exists for plotting points and functions in polar coordinates, similar to Figure 6.4.4.

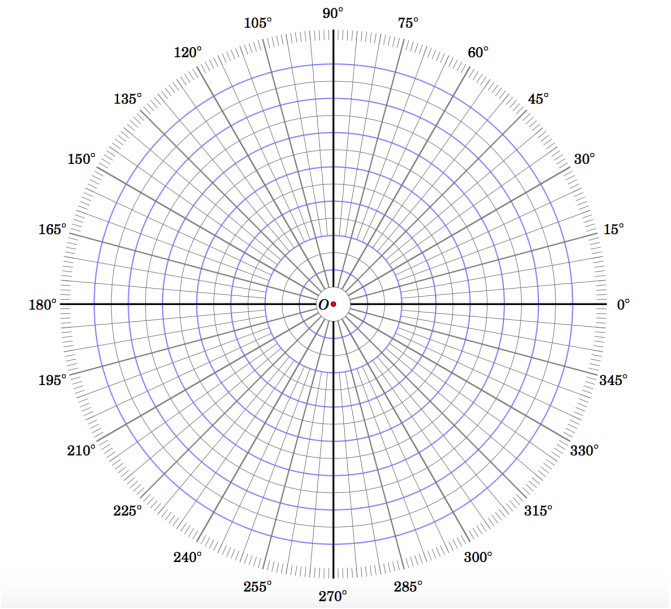

FIGURE 6.4.4 Polar Coordinate Graph

The angle θ can be given in either degrees or radians, whichever is more convenient. Radians are often preferred when graphing functions in polar coordinates. The reason is that, unlike degrees, radians can be considered "unitless" (as we mentioned in Chapter 4). This is desirable when a function given in polar coordinates is expressed as r as a function of θ (similar to how, in Cartesian coordinates (x, y), functions are usually expressed as y as a function of x). For example, if a function in polar coordinates is written as $r = 2\,\theta$, then r would have the same units as θ. But r should be a unitless quantity, hence using radians for θ makes more sense in this case.

Example 6.14

Express the spiral from Figure 6.4.1 in polar coordinates.

Solution:

We will use radians for θ. The goal is to find some equation involving r and θ that describes the spiral. We see that

$$\begin{aligned} \theta = 0 \quad &\Rightarrow \quad r = 1 \\ \theta = 2\pi \quad &\Rightarrow \quad r = 2 \\ \theta = 4\pi \quad &\Rightarrow \quad r = 3 \\ &\vdots \\ \theta = 2\pi\,k \quad &\Rightarrow \quad r = 1 + k \end{aligned}$$

for $k = 0, 1, 2, \ldots$. In fact, that last relation holds for any nonnegative real number k (why?). So for any $\theta \geq 0$,

$$\theta = 2\pi\,k \quad \Rightarrow \quad k = \frac{\theta}{2\pi} \quad \Rightarrow \quad r = 1 + k = 1 + \frac{\theta}{2\pi}\,.$$

Hence, the spiral can be written as $\boxed{r = 1 + \frac{\theta}{2\pi}}$ for $\theta \geq 0$. The graph is shown in Figure 6.4.5, along with the Gnuplot commands to create the graph.

```
set polar
set size square
set samples 2000
unset key
set zeroaxis
set xlabel "x"
set ylabel "y"
plot [0:6*pi] 1 + t/(2*pi)
```

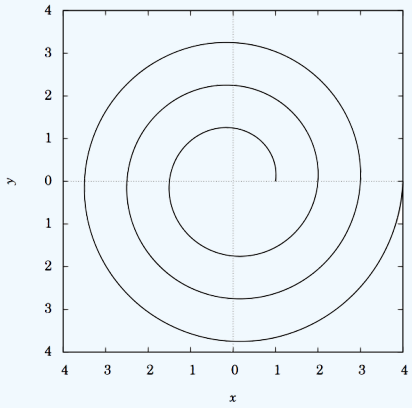

FIGURE 6.4.5 $r = 1 + \frac{\theta}{2\pi}$

Note that when using the `set polar` command, Gnuplot will assume that the function being plotted is r as a function of θ (represented by the variable `t` in Gnuplot).

Figure 6.4.6 shows how to convert between polar coordinates and Cartesian coordinates. For a point with polar coordinates (r, θ) and Cartesian coordinates (x, y):

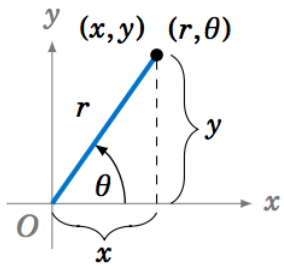

FIGURE 6.4.6

Polar to Cartesian:

$$\boxed{x = r\cos\theta \qquad y = r\sin\theta} \tag{6.4.1}$$

Cartesian to Polar:

$$\boxed{r = \pm\sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x} \text{ if } x \neq 0} \tag{6.4.2}$$

Note that in Equation 6.4.2, if $x = 0$ then $\theta = \pi/2$ or $\theta = 3\pi/2$. Also, if $x \neq 0$ and $y \neq 0$ then the two possible solutions for θ in the equation $\tan\theta = \frac{y}{x}$ are in opposite quadrants (for $0 \leq \theta < 2\pi$). If the angle θ is in the same quadrant as the point (x, y), then $r = \sqrt{x^2 + y^2}$ (i.e. r is positive); otherwise $r = -\sqrt{x^2 + y^2}$ (i.e. r is negative).

Example 6.15

Convert the following points from polar coordinates to Cartesian coordinates:
(a) $(2, 30^\circ)$; **(b)** $(3, 3\pi/4)$; **(c)** $(-1, 5\pi/3)$

Solution:
(a) Using Equation 6.4.1 with $r = 2$ and $\theta = 30^\circ$, we get:

$$(x, y) = (r\cos\theta, r\sin\theta) = (2\cos 30^\circ, 2\sin 30^\circ) = \left(2 \cdot \tfrac{\sqrt{3}}{2}, 2 \cdot \tfrac{1}{2}\right) \Rightarrow \boxed{(x, y) = (\sqrt{3}, 1)}$$

(b) Using Equation 6.4.1 with $r = 3$ and $\theta = 3\pi/4$, we get:

$$(x, y) = (r\cos\theta, r\sin\theta) = \left(3\cos\tfrac{3\pi}{4}, 3\sin\tfrac{3\pi}{4}\right) = \left(3\cdot\tfrac{-1}{\sqrt{2}}, 3\cdot\tfrac{1}{\sqrt{2}}\right) \Rightarrow \boxed{(x, y) = \left(\tfrac{-3}{\sqrt{2}}, \tfrac{3}{\sqrt{2}}\right)}$$

(c) Using Equation 6.4.1 with $r = -1$ and $\theta = 5\pi/3$, we get:

$$(x, y) = (r\cos\theta, r\sin\theta) = \left(-1\cos\tfrac{5\pi}{3}, -1\sin\tfrac{5\pi}{3}\right) = \left(-1\cdot\tfrac{1}{2}, -1\cdot\tfrac{-\sqrt{3}}{2}\right) \Rightarrow$$

$$\boxed{(x, y) = \left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right)}$$

Example 6.16

Convert the following points from Cartesian coordinates to polar coordinates:
(a) $(3, 4)$; **(b)** $(-5, -5)$

Solution:
(a) Using formula 6.4.2 with $x = 3$ and $y = 4$, we get:

$$\tan\theta = \frac{y}{x} = \frac{4}{3} \quad \Rightarrow \quad \theta = 53.13^\circ \quad \text{or} \quad \theta = 233.13^\circ$$

Since $\theta = 53.13^\circ$ is in the same quadrant (QI) as the point $(x, y) = (3, 4)$, we can take $r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = 5$. Thus, $\boxed{(r, \theta) = (5, 53.13^\circ)}$.

Note that if we had used $\theta = 233.13^\circ$, then we would have $(r, \theta) = (-5, 233.13^\circ)$.

(b) Using Equation 6.4.2 with $x = -5$ and $y = -5$, we get:

$$\tan\theta = \frac{y}{x} = \frac{-5}{-5} = 1 \quad \Rightarrow \quad \theta = 45^\circ \quad \text{or} \quad \theta = 225^\circ$$

Since $\theta = 225^\circ$ is in the same quadrant (QIII) as the point $(x, y) = (-5, -5)$, we can take $r = \sqrt{x^2 + y^2} = \sqrt{(-5)^2 + (-5)^2} = 5\sqrt{2}$. Thus, $\boxed{(r, \theta) = (5\sqrt{2}, 225^\circ)}$.

\noindent Note that if we had used $\theta = 45^\circ$, then we would have $(r, \theta) = (-5\sqrt{2}, 45^\circ)$.
\end{exmp}

Example 6.17

Write the equation $x^2 + y^2 = 9$ in polar coordinates.

Solution:
This is just the equation of a circle of radius 3 centered at the origin. Since $r = \pm\sqrt{x^2 + y^2} = \pm\sqrt{9}$, in polar coordinates the equation can be written as simply $\boxed{r = 3}$.

Example 6.18

Write the equation $x^2 + (y - 4)^2 = 16$ in polar coordinates.

Solution: This is the equation of a circle of radius 4 centered at the point $(0, 4)$. Expanding the equation, we get:

$$\begin{aligned} x^2 + (y-4)^2 &= 16 \\ x^2 + y^2 - 8y + 16 &= 16 \\ x^2 + y^2 &= 8y \\ r^2 &= 8r\sin\theta \\ r &= 8\sin\theta \end{aligned}$$

Why could we cancel r from both sides in the last step? Because the point $(0,0)$ is on the circle, canceling r does not eliminate $r=0$ as a potential solution of the equation (since $\theta = 0^\circ$ would make $r = 8\sin\theta = 8\sin 0^\circ = 0$). Thus, the equation is $\boxed{r = 8\sin\theta}$.

Example 6.19

Write the equation $y = x$ in polar coordinates.

Solution:
This is the equation of a line through the origin. So when $x = 0$, we know that $y = 0$. When $x \neq 0$, we get:

$$\begin{aligned} y &= x \\ \frac{y}{x} &= 1 \\ \tan\theta &= 1 \\ \theta &= 45^\circ \end{aligned}$$

Since there is no restriction on r, we could have $r = 0$ and $\theta = 45^\circ$, which would take care of the case $x = 0$ (since then $(x, y) = (0, 0)$, which is the same as $(r, \theta) = (0, 45^\circ)$). Thus, the equation is $\boxed{\theta = 45^\circ}$.

Example 6.20

Prove that the distance d between two points (r_1, θ_1) and (r_2, θ_2) in polar coordinates is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)} \ .$$

Solution:
The idea here is to use the distance formula in Cartesian coordinates, then convert that to polar coordinates. So write

$$\begin{aligned} x_1 &= r_1\cos\theta_1 \qquad y_1 = r_1\sin\theta_1 \\ x_2 &= r_2\cos\theta_2 \qquad y_2 = r_2\sin\theta_2 \ . \end{aligned}$$

Then (x_1, y_1) and (x_2, y_2) are the Cartesian equivalents of (r_1, θ_1) and (r_2, θ_2), respectively. Thus, by the Cartesian coordinate distance formula,

$$\begin{aligned} d^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= (r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2 \\ &= r_1^2\cos^2\theta_1 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_2^2\cos^2\theta_2 + r_1^2\sin^2\theta_1 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_2^2\sin^2\theta_2 \\ &= r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\ d^2 &= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)\ , \end{aligned}$$

so the result follows by taking square roots of both sides.

In Example 6.17 we saw that the equation $x^2+y^2=9$ in Cartesian coordinates could be expressed as $r=3$ in polar coordinates. This equation describes a circle centered at the origin, so the circle is symmetric about the origin. In general, polar coordinates are useful in situations when there is symmetry about the origin (though there are other situations), which arise in many physical applications.

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6.E: Additional Topics (Exercises)

These are homework exercises to accompany Corral's "Elementary Trigonometry" Textmap. This is a text on elementary trigonometry, designed for students who have completed courses in high-school algebra and geometry. Though designed for college students, it could also be used in high schools. The traditional topics are covered, but a more geometrical approach is taken than usual. Also, some numerical methods (e.g. the secant method for solving trigonometric equations) are discussed.

6.1 Exercise

For Exercises 1-12, solve the given equation (in radians).

6.1.1 $\tan\ \theta + 1 = 0$

6.1.2 $2\ \cos\ \theta + 1 = 0$

6.1.3 $\sin\ 5\theta + 1 = 0$

6.1.4 $2\ \cos^2\ \theta - \sin^2\ \theta = 1$

6.1.5 $2\ \sin^2\ \theta - \cos\ 2\theta = 0$

6.1.6 $2\ \cos^2\ \theta + 3\ \sin\ \theta = 0$

6.1.7 $\cos^2\ \theta + 2\ \sin\ \theta = -1$

6.1.8 $\tan\ \theta + \cot\ \theta = 2$

6.1.9 $\sin\ \theta = \cos\ \theta$

6.1.10 $2\ \sin\ \theta - 3\ \cos\ \theta = 0$

6.1.11 $\cos^2\ 3\theta - 5\ \cos\ 3\theta + 4 = 0$

6.1.12 $3\ \sin\ \theta - 4\ \cos\ \theta = 1$

6.2 Exercise

6.2.1 One obvious solution to the equation $2\ \sin\ x = x$ is $x = 0$. Write a program to find the other solution(s), accurate to at least within 1.0×10^{-20}. You can use any programming language, though you may find it easier to just modify the code in Listing 6.1 (only one line needs to be changed!). It may help to use Gnuplot to get an idea of where the graphs of $y = 2\ \sin\ x$ and $y = x$ intersect.

6.2.2 Repeat Exercise 1 for the equation $\sin\ x = x^2$.

6.2.3 Use Octave or some other program to find the maximum and minimum of $y = \cos\ 5x - \sin\ 3x$.

6.3 Exercise

For Exercises 1-16, calculate the given expression.

6.3.1 $(2+3i)\ +\ (-3-2i)$

6.3.2 $(2+3i)\ -\ (-3-2i)$

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6.3.3 $(2+3i)\cdot(-3-2i)$

6.3.4 $(2+3i)/(-3-2i)$

6.3.5 $\overline{(2+3i)}+\overline{(-3-2i)}$

6.3.6 $\overline{(2+3i)}-\overline{(-3-2i)}$

6.3.7 $(1+i)/(1-i)$

6.3.8 $|-3+2i|$

6.3.9 i^3

6.3.10 i^4

6.3.11 i^5

6.3.12 i^6

6.3.13 i^7

6.3.14 i^8

6.3.15 i^9

6.3.16 i^{2009}

For Exercises 17-24, prove the given identity for all complex numbers.

6.3.17 $\overline{(\overline{z})}=z$

6.3.18 $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$

6.3.19 $\overline{z_1-z_2}=\overline{z_1}-\overline{z_2}$

6.3.20 $\overline{z_1\,z_2}=\overline{z_1}\,\overline{z_2}$

6.3.21 $\overline{\left(\dfrac{z_1}{z_2}\right)}=\dfrac{\overline{z_1}}{\overline{z_2}}$

6.3.22 $|z|=|\overline{z}|$

6.3.23 $|z_1\,z_2|=|z_1|\,|z_2|$

6.3.24 $\left|\dfrac{z_1}{z_2}\right|=\dfrac{|z_1|}{|z_2|}$

For Exercises 25-30, put the given number in trigonometric form.

6.3.25 $2+3i$

6.3.26 $-3-2i$

6.3.27 $1-i$

6.3.28 $-i$

6.3.29 1

6.3.30 -1

6.3.31 Verify that De Moivre's Theorem holds for the power $n=0$.

For Exercises 32-35, calculate the given number.

6.3.32 $3\left(\cos\ 14^{\circ}\ +i\ \sin\ 14^{\circ}\right)\cdot\ 2\left(\cos\ 121^{\circ}\ +i\ \sin\ 121^{\circ}\right)$

6.3.33 $\left[3\left(\cos\ 14^{\circ}\ +i\ \sin\ 14^{\circ}\right)\right]^{4}$

6.3.34 $\left[3\left(\cos\ 14^{\circ}\ +i\ \sin\ 14^{\circ}\right)\right]^{-4}$

6.3.35 $\dfrac{3\left(\cos\ 14^{\circ}\ +i\ \sin\ 14^{\circ}\right)}{2\left(\cos\ 121^{\circ}\ +i\ \sin\ 121^{\circ}\right)}$

6.3.36 Find the three cube roots of $-i$.

6.3.37 Find the three cube roots of $1+i$.

6.3.38 Find the three cube roots of 1.

6.3.39 Find the three cube roots of -1.

6.3.40 Find the five fifth roots of 1.

6.3.41 Find the five fifth roots of -1.

6.3.42 Find the two square roots of $-2+2\sqrt{3}\,i$.

6.3.43 Prove that if z is an n^{th} root of a real number a, then so is $\bar{z}$.(*Hint: Use Exercise 20.*)

6.4 Exercise

For Exercises 1-5, convert the given point from polar coordinates to Cartesian coordinates.

6.4.1 $(6,210^{\circ})$

6.4.2 $(-4,3\pi)$

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6.4.3 $(2, 11\pi/6)$

6.4.4 $(6, 90^\circ)$

6.4.5 $(-1, 405^\circ)$

For Exercises 6-10, convert the given point from Cartesian coordinates to polar coordinates.

6.4.6 $(3, 1)$

6.4.7 $(-1, -3)$

6.4.8 $(0, 2)$

6.4.9 $(4, -2)$

6.4.10 $(-2, 0)$

For Exercises 11-18, write the given equation in polar coordinates.

6.4.11 $(x-3)^2 + y^2 = 9$

6.4.12 $y = -x$

6.4.13 $x^2 - y^2 = 1$

6.4.14 $3x^2 + 4y^2 - 6x = 9$

6.4.15 Graph the function $r = 1 + 2\cos\theta$ in polar coordinates.

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